Fractional Fourier Transform, Scale Operator, and Uncertainty Relations

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Abstract

The Implications of a fractional (angular) Fourier transformation on canonical communication relations, uncertainty relations, and scaling are derived.
It is well known [1] that Fourier Transform (F.T) plays a fundamental role in quantum mechanics. If $\hat{q}$ and $\hat{p}$ are the position and momentum self-adjoint operators which satisfy the canonical communication relations\(^*\), [CCR]

$$[\hat{q}, \hat{p}] = i, \quad (1)$$

their bases are related through the kernel of F.T.,

$$k(q, p) \equiv \langle q | p \rangle \frac{1}{\sqrt{2\pi}} e^{-iqp}, \quad (2)$$

Indeed, $k^*(q, p)$ furnish a basis for $p \left( = -i \frac{q}{\partial q} \right)$,

$$\hat{p} k^*(q, p) = p k^*(q, p), \quad (3)$$

A fractional F.T. (fr. F.T.) generalizes (1) and is defined by the kernel [2]

$$\langle q | p_\alpha \rangle = k_\alpha(q, p) = \sqrt{\frac{1 - i \cot(\alpha)}{2\pi}} \exp i \left\{ \frac{q^2 + p^2}{2} \cot(\alpha) - qpcosec(\alpha) \right\}, \quad \alpha \neq \text{multiple of } 2\pi,$$

$$= \delta(q - p), \quad \alpha = 2n\pi, \quad n = \text{integer},$$

$$= \delta(q + p), \quad \alpha = (2n + 1)\pi \quad (4)$$

It follows that

$$k_{\frac{\alpha}{2}}(q, p) = k(q, p), \quad (5)$$

and

$$|\langle q | p_\alpha \rangle| = \sqrt{\frac{\cosec(\alpha)}{2\pi}}, \quad (6)$$

The object of this note is to derive some consequences that follow in replacing F.T. by fr. F.T. in quantum mechanics, especially the CCR, Heisenberg and Robertson-Schrödinger uncertainty relations [3], and scaling operators [4].

\(^*\)We use the units $\hbar = c = 1$
From the definition of the fr. F.T., it is easily verified that

\[ \hat{p}_\alpha k^*_\alpha(q, p) = p k^*_\alpha(q, p) \]  

(7)

where the "fractional" momentum operator \( \hat{p}_\alpha \) is

\[ \hat{p}_\alpha \equiv \cos \alpha \hat{q} + \sin \alpha \hat{p} \]  

(8)

Equation 8 has been interpreted[2] as a rotation in the \((\hat{q}, \hat{p})\) space. If it is true, \( \hat{q} \) will be rotated to

\[ \hat{q}_\alpha \equiv \sin \alpha \hat{q} - \cos \alpha \hat{p} \]  

(9)

and of course, CCR will remain invariant. On the other hand, if \( \hat{p}_\alpha \) is interpreted as the fractional \( \hat{p} \) from the direction of \( \hat{q} \) \( [\alpha = \text{angle between } \hat{q} \text{ and } \hat{p}_\alpha] \), it follows that

\[ [\hat{q}, \hat{p}_\alpha] = i \sin \alpha, \]  

(10)

and

\[ [\hat{p}_\alpha, \hat{p}_\beta] = i \sin (\beta - \alpha), \]  

(11)

The Heisenberg uncertainty relations will be modified to

\[ (\Delta q)^2 (\Delta p_\alpha)^2 = \frac{1}{4} \sin^2 \alpha, \]  

(12)

where, as usual,

\[ (\Delta q)^2 (\Delta x)^2 \equiv \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2, \]  

(13)

The modified Robertson-Schrödinger [R.S.] uncertainty relations reads as

\[ \det \begin{vmatrix} (\Delta q)^2 & \Delta_\alpha(q, p) \\ \Delta_\alpha(q, p) & (p_\alpha)^2 \end{vmatrix} \geq \frac{1}{4} \sin^2(\alpha), \]  

(14)

where

\[ \Delta_\alpha(q, p) \equiv \left| \frac{1}{2} \left\{ \Delta \hat{q}, \Delta \hat{p}_\alpha \right\} \right| \]

\[ \Delta \hat{x} \equiv \hat{x} - \langle \hat{x} \rangle \]  

(15)
Here, the bracket \( \{ \} \) stands for the anticommutator. For \( \alpha = \frac{\pi}{4} \),

\[
\Delta_\alpha(q, p) \rightarrow \Delta(q, p) = \{ \Delta \hat{q}, \Delta \hat{p}_\alpha \}_+
\]  

(16)

and Equation 14 will result in the standard (R.S) uncertainty relation which is stronger than the Heisenberg uncertainty relation. Expanding Equation 14, one finds

\[
(\Delta q)^2 (\Delta p_\alpha)^2 \geq \frac{1}{4} \sin^2 \alpha + \frac{1}{4} \left| \langle \sin \alpha \{ \Delta \hat{q}, \Delta \hat{p}_\alpha \}_+ + 2 \cos \alpha (\Delta q^2) \rangle \right|^2
\]  

(17)

Also,

\[
(\Delta p_\alpha)^2 (\Delta p_\beta)^2 \geq \frac{1}{4} \sin^2 (\alpha - \beta) + \frac{1}{4} \left| \langle 2 \sin \alpha \sin \beta (\Delta q)^2 + 2 \cos \alpha \cos \beta (\Delta q)^2 + \sin (\alpha + \beta) \{ \Delta \hat{q}, \Delta \hat{p}_\alpha \}_+ \rangle \right|^2
\]  

(18)

Some of these relations have been notified earlier [5]. The Scale Operator (generator for squeezing)

\[
\hat{X} \equiv \frac{1}{2} \{ \hat{q}, \hat{p} \}
\]  

(19)

will get modified to

\[
\hat{X}_\alpha \equiv \frac{1}{2} \{ \hat{q}, \hat{p}_\alpha \} = \hat{X} \sin \alpha + q^2 \cos \alpha
\]  

(20)

The squeezing operator [6], \( \exp^{\zeta \hat{X}} (\zeta = \text{squeezing state parameter}) \) will be modified as

\[
\hat{S}_\alpha \equiv \exp^{\zeta \hat{X}_\alpha}
\]  

(21)

Defining the unitary Weyl operators [7],

\[
\hat{U}_\alpha \equiv \exp^{i \zeta \hat{p}_\alpha}, \zeta = \text{real}
\]

\[
\hat{V} \equiv \exp^{i \eta \hat{p}}, \eta = \text{real}
\]  

(22)

Equation 10 implies

\[
\hat{U}_\alpha \hat{V} = \exp^{i \zeta \eta \sin \alpha} \hat{V} \hat{U}_\alpha
\]  

(23)
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