LTI Systems & Fourier Series

It can be easily shown that a *linear time-invariant* (LTI) map, $y(t) = \mathbf{L}(x(t))$, with input $x(t)$ and output $y(t)$ satisfies a non-homogeneous, linear, constant coefficient, differential equation of the form

$$\sum_{k=0}^{N} p_k y^{(k)}(t) \;=\; \sum_{m=0}^{M} q_m x^{(m)}(t) \;\;,\;\; y^{(k)}(t) \;\equiv\; \frac{d^k y}{dt^k}.$$

The natural response of this system of course is determined by setting the input $x(t) = 0$ and assuming that the solution to system is of the form $y_h(t) = \exp(\lambda t)$. The solution to the homogeneous part is then given by

$$y_h(t) \;=\; \sum_{k=1}^{N} f_k \exp(\lambda_k t)\,,$$

where λ_k are the solutions to the characteristic equation

$$\sum_{k=0}^{N} p_k \lambda^k = 0.$$

The particular solution is obtained by setting the input to $x(t) = \exp(j\omega_0 t)$ and assuming that the output is of the form:

$$y_{par}(t) \;=\; \mathbf{L}(\exp(j\omega_0 t)) \;=\; H(\omega_0)\exp(j\omega_0 t)\,.$$

The goal is to determine the complex constant $H(\omega)$, which is also the eigenvalue of the linear map $\mathbf{L}$ corresponding to the eigenfunction $x(t) = \exp(j\omega t)$. By substituting this solution into the differential equation system we determine the complex eigenvalue for a general sinusoidal frequency ω as :

$$H(\omega) \;=\; \frac{\sum_{k=0}^{M} q_k (j\omega)^k}{\sum_{k=0}^{N} p_k (j\omega)^k} \;=\; \frac{Q(\omega)}{P(\omega)} \;=\; \left(\frac{|Q(\omega)|}{|P(\omega)|}\right) \exp\left(j\left(\arg(Q(\omega)) - \arg(P(\omega))\right)\right).$$

This ratio is refered to as the *frequency response* of the system $\mathbf{L}$. Since, the system is an LTI system we can use the superposition principle to determine the output when the input to the system is a periodic signal:

$$x(t) \;=\; \sum_{k=-\infty}^{\infty} c_k \exp(jk\omega_0 t)\,.$$

The output of the LTI system for this periodic input signal is of the form

$$y(t) \;=\; \sum_{k=-\infty}^{\infty} \underbrace{H(k\omega_0)c_k}_{d_k} \exp(jk\omega_0 t)\,.$$

This implies that the output is also periodic with the same fundamental frequency ω_0 and that the Fourier coefficients of the output are given by $d_k \;=\; H(k\omega_0)c_k$.