ECE-314, Fall 08

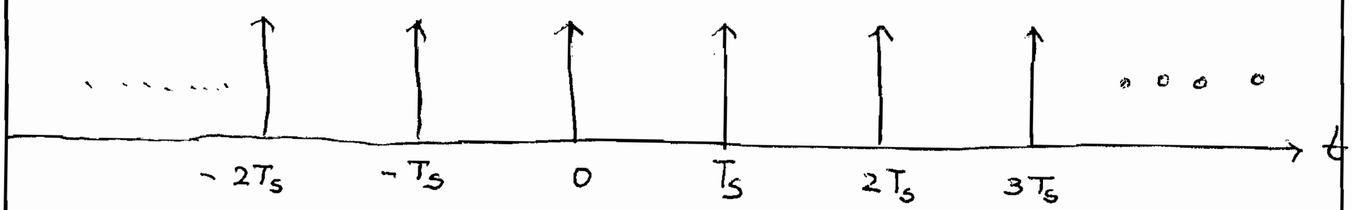
Signals and Systems

Example: Fourier Series

Consider the periodic impulse train signal

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s),$$

where T_s is the teeth spacing



- This signal is obviously periodic with fundamental period $T_0 = T_s$

- This means $x(t)$ has a Fourier Series representation given by:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left\{ j \frac{2\pi}{T_0} n t \right\}$$

$$C_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} x(t) \exp\left\{-jk\frac{2\pi}{T_s}t\right\} dt$$

$$x(t) = \delta(t), \quad t \in \left[-\frac{T_s}{2}, \frac{T_s}{2}\right]$$

$$\Rightarrow \quad C_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) \exp\left\{-jk\frac{2\pi}{T_s}t\right\} dt$$

By the sifting theorem:

$$\delta(t) \exp\left\{-jk\frac{2\pi}{T_s}t\right\} = \exp\left\{-jk\frac{2\pi}{T_s}\cdot 0\right\} \delta(t)$$
$$= \delta(t)$$

$$\Rightarrow \quad C_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t)\, dt = \frac{1}{T_s} \quad \forall k$$

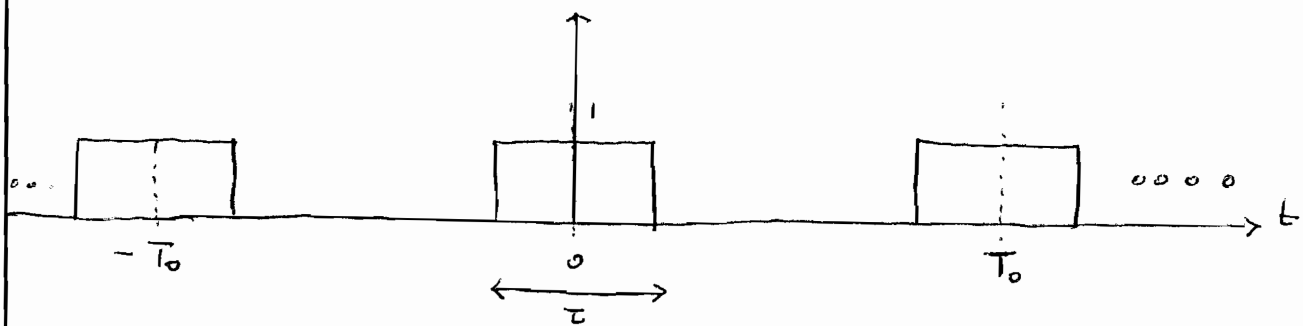
$$\Rightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} \exp\left\{jk\frac{2\pi}{T_s}t\right\}$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \exp\left\{jk\frac{2\pi}{T_s}t\right\}$$

This is a signal that has an infinite number of harmonics at $k\frac{2\pi}{T_s}$ all of them with Equal strength (all-pass signal)

Consider how the periodic square pulse of duty cycle $\frac{\tau}{T_0}$ given by:

$$x(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - nT_0}{\tau}\right)$$



In this case:

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \exp\left(-jk\frac{2\pi}{T_0} t\right) dt$$

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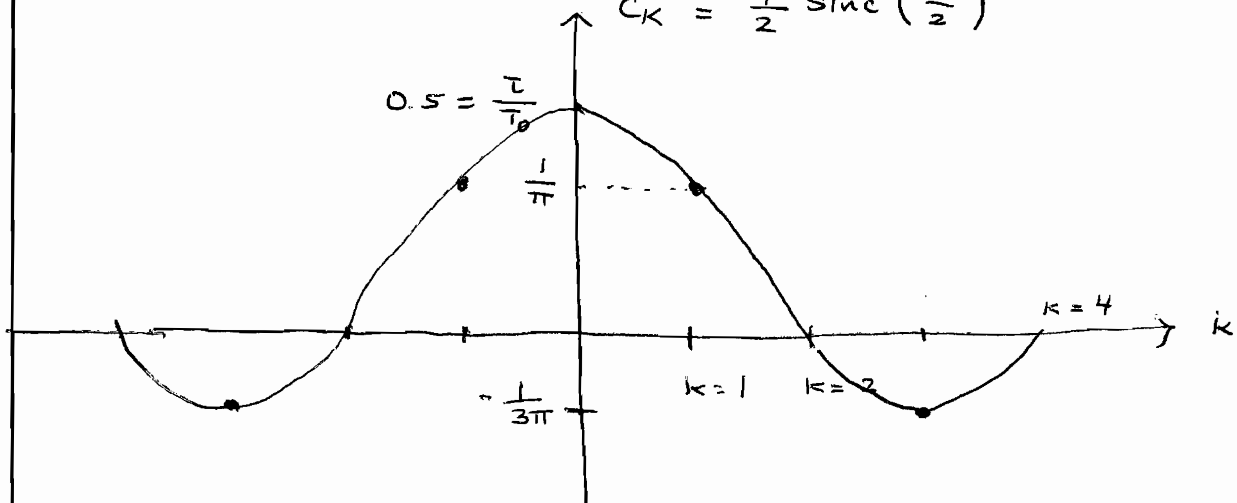
$$C_k = \frac{1}{\cancel{T_0}} \left. \frac{\exp\left\{-jk\frac{2\pi}{T_0} t\right\}}{-jk\frac{2\pi}{\cancel{T_0}}} \right|_{-\tau/2}^{\tau/2}$$

$$C_k = \frac{-1}{2jk\pi} \left\{ \exp\left\{-jk\frac{2\pi}{T_0}\frac{\tau}{2}\right\} - \exp\left\{jk\frac{2\pi}{T_0}\frac{\tau}{2}\right\} \right\}$$

$$C_k = \frac{1}{\cancel{(-2j)}(k\pi)} \cancel{(-2j)} \sin\left(\frac{k\pi\tau}{T_0}\right)$$

$$C_k = \left(\frac{\tau}{T_0}\right) \operatorname{sinc}\left(\frac{k\tau}{T_0}\right)$$

$$C_K = \frac{1}{2} \operatorname{sinc}\left(\frac{k}{2}\right)$$



Observations:

- Harmonics at higher frequencies have fading strength (Lowpass signal)
- Harmonics at $k = 2, 4, 6 \ldots$ have no strength (absent!)
- In general zero-crossings are at

$$k\cancel{\pi}\frac{\tau}{T_0} = r\cancel{\pi}, \quad r \in I$$

or $$k = \frac{r}{\frac{\tau}{T_0}}, \quad r \in I$$