Example: Bilateral Laplace Inversion

Consider a continuous-time signal whose bilateral Laplace transform is given by the expression:

$$X(s) = \frac{s+1}{(s+2)(s+3)(s-1)}.$$

This expression has one zero at $s = -1$ and three poles at $s = -2, -3, 1$. Performing a partial fractions expansion on this expression because it is purely rational:

$$X(s) = \frac{k_1}{s+2} + \frac{k_2}{s+3} + \frac{k_3}{s-1}.$$

Comparing coefficients or substituting specific values of s we obtain:

$$X(s) = \frac{1/3}{s+2} + \frac{-1/2}{s+3} + \frac{1/6}{s-1}$$

By definition the ROC of this expression cannot contain singularities. Therefore there are several possibilities for the ROC of this expression:

1. $\sigma < -3$: Region I
2. $\sigma > 1$: Region II
3. $-2 < \sigma < 1$: Region III

Note that of these three possibilities only Region III includes the imaginary axis and therefore will result in a absolutely integrable inverse. Before we begin to investigate the inverse, we will make use of the following Laplace transform pairs:

$$\begin{aligned} \mathcal{L}\left(e^{-at}u(t)\right) &= \frac{1}{s+a},\ \sigma > -a, \\ \mathcal{L}\left(e^{at}u(-t)\right) &= \frac{1}{a-s},\ \sigma < a. \end{aligned}$$

We will also make use of the pairs obtained by replacing a with $-a$:

$$\begin{aligned} \mathcal{L}\left(e^{at}u(t)\right) &= \frac{1}{s-a},\ \sigma > a, \\ \mathcal{L}\left(-e^{-at}u(-t)\right) &= \frac{1}{s+a}, \sigma < -a. \end{aligned}$$

The solution for the Laplace inverse in this case is given by:

$$\begin{aligned} x(t) &= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+2}, \sigma > -2\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+3}, \sigma > -3\right) \\ &+ \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s-1}, \sigma < 1\right). \end{aligned}$$

Upon using the appropriate Laplace transform pairs we obtain:

$$x(t) = \frac{1}{3}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t) + \frac{1}{6}e^{t}u(-t),$$

In region I, i.e., $\sigma < -3$, we obtain the inverse:

$$\begin{aligned} x(t) &= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+2}, \sigma < -2\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+3}, \sigma < -3\right) \\ &+ \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s-1}, \sigma < 1\right). \end{aligned}$$

Using the pairs described before:

$$x(t) = -\frac{1}{3}e^{-2t}u(-t) + \frac{1}{2}e^{-3t}u(-t) - \frac{1}{6}e^{-t}u(-t).$$

Clearly this solution produces a non-causal signal and will not be bounded as $t \to -\infty$. For region II, we have the solution:

$$\begin{aligned} x(t) &= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+2}, \sigma > -2\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+3}, \sigma > -3\right) \\ &+ \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s-1}, \sigma > 1\right). \end{aligned}$$

The corresponding solution for the inverse is:

$$x(t) = \frac{1}{3}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t) + \frac{1}{6}e^{t}u(t).$$

Again this solution is purely causal but will not be bounded as $t \to \infty$.