System Impulse Response

Uptill now we have looked at a system as just a black box that processes an input signal $x(t)$ to produce an output signal $y(t)$. Our goal here is to qualify what is in the black box when the system $\mathbf{L}$ is *linear time-invariant* (LTI).

The story begins with the linear sifting integral expression for the input signal in terms of Dirac impulse functions. First we look at the sampling property of the impulse function:

$$x(t)\delta(t-\tau) \quad \equiv \quad x(t)\lim_{\in\to 0}\Pi_{\in}(t-\tau) \quad = \quad \lim_{\in\to 0} x(t)\Pi_{\in}(t-\tau),$$

where the function $r_{\in}(t)$ is defined by

$$\Pi_{\in}(t) = \begin{cases} \left(\frac{1}{\in}\right) & -\frac{\in}{2} \le t \le \frac{\in}{2} \\ \\ 0 & \text{otherwise.} \end{cases}$$

If the pulsewidth, $\in$, of the function $r_{\in}(t)$ is chosen small enough so that the signal x(t) is approximately a constant over this interval then the product can be written as

$$x(t)\delta(t-\tau) \quad = \quad \lim_{\in\to 0} x(\tau)\Pi_{\in}(t-\tau) \quad = \quad x(\tau)\delta(t-\tau).$$

In a similar fashion, by looking at the product $x(t)\delta(t-k\Delta)$ in the limit where Δ, the sampling interval goes to zero, we can completely express the input signal $x(t)$ interms of delta functions as:

$$x(t) \quad = \quad \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)\Pi_{\Delta}(t-k\Delta)\Delta \quad = \quad \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

The action of the map $\mathbf{L}$ on this input signal $x(t)$: $y(t) = \mathbf{L}(x(t))$ is given by

$$y(t) \quad = \quad \mathbf{L}\left(\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right).$$

Since integration is a linear operation and since the map is a linear map, the integration and the system operator $\mathbf{L}$ can be exchanged to obtain

$$y(t) \quad = \quad \int_{-\infty}^{\infty} x(\tau)\mathbf{L}\left(\delta(t-\tau)\right)d\tau.$$

We denote the response of the LTI system to a Dirac impulse input $x(t) = \delta(t)$ as $h(t) = \mathbf{L}(\delta(t))$. This is also called as the system impulse response, i.e., the output of the LTI system when the input is a Dirac delta function. Since the system in addition to being linear is also time-invariant, the output of the system to the delayed impulse function is $h(t-\tau) \quad = \quad \mathbf{L}(x(t-\tau))$. Incorporating this into the output we see that the output of the system $\mathbf{L}$ to a general input signal $x(t)$ is given by

$$y(t) \quad = \quad \mathbf{L}(x(t)) \quad = \quad \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad = \quad x(t) * h(t).$$

This important input-output relation is refered to as the convolution theorem. The notation $*$ denotes the convolution integral in the above equation. This relation describes completely the time-domain input output characteristics of the LTI system.

The Convolution Integral

In general, one can define the continuous-time convolution operation between two signals $x_1(t)$ and $x_2(t)$ as

$$x_1(t) * x_2(t) \equiv \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau.$$

1. Commutative: This linear convolution integral with the substitution of variables $\sigma = t - \tau$ can be rewritten as

$$\begin{aligned} y(t) &= x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x_1(t-\sigma)x_2(\sigma)d\sigma = x_2(t) * x_1(t). \end{aligned}$$

2. Associative:

$$(x_1(t) * x_2(t)) * x_3(t) = x_1(t) * (x_2(t) * x_3(t)).$$

3. Distributive:

$$x_1(t) * (x_2(t) + x_3(t)) = x_1(t) * x_2(t) + x_1(t) * x_3(t).$$

4. Identity element: With the convolution operation defined by the integral the sifting relation that was discussed before can be written as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t).$$

   In other words, the identity element over the binary operation of continuous-time convolution is the Dirac delta function.

5. Inverse element: The inverse element of a LTI system defined by its impulse response, $h(t)$, over the binary operation of convolution is the LTI system with impulse response $h_I(t)$ that satisfies

$$h(t) * h_I(t) = \int_{-\infty}^{\infty} h(\tau)h_I(t-\tau)d\tau = \delta(t).$$