ECE-314, Fall 2018
Signals & Systems

Example: Fourier Series Representation

$$\phi_k(t) = \exp\left\{ j \frac{2\pi}{T_0} k t \right\}$$

$$-\infty \le k \le \infty, \quad 0 \le t \le T_0$$

$$\langle \phi_p(t), \phi_q(t) \rangle = \int_0^{T_0} \phi_p(t)\, \phi_q^*(t)\, dt$$

$$= \int_0^{T_0} \exp\left\{ j \frac{2\pi}{T_0} p t \right\} \exp\left(-j \frac{2\pi}{T_0} q t \right) dt$$

$$= \int_0^{T_0} \exp\left\{ j \frac{2\pi}{T_0} (p-q) t \right\} dt$$

$$= \frac{\exp\left\{ j \frac{2\pi}{T_0} (p-q) t \right\} \Big|_0^{T_0}}{j \frac{2\pi}{T_0} (p-q)}$$

$$= \frac{1}{j \frac{2\pi}{T_0} (p-q)} \left\{ e^{j \frac{2\pi}{T_0} (p-q) T_0} - 1 \right\}$$

$$= 0, \quad p \ne q$$

$$\langle \phi_p(t), \phi_p(t) \rangle = \int_0^{T_0} (1)\,dt = T_0$$

$$\langle \phi_p(t), \phi_q(t) \rangle = \begin{cases} T_0, & p = q \\ 0, & p \neq q \end{cases}$$

$\Rightarrow \{\phi_p(t)\}_{p=-\infty}^{\infty}$ constitutes a pair-wise orthogonal sequence

From notes in class

$$c[k] = \frac{\langle x(t), \phi_k(t) \rangle}{\langle \phi_k(t), \phi_k(t) \rangle}$$

$$= \frac{1}{T_0} \int_0^{T_0} x(t) \exp\left(-j \frac{2\pi}{T_0} k t\right) dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} c[k] \exp\left(j \frac{2\pi}{T_0} k t\right)$$

In this case : $a = 0$, $b = T_0$

The signal space is all continuous periodic signals.

Examples:

(a) $$x(t) = \cos(\omega_0 t) = \cos\left(\frac{2\pi}{T_0} t\right), \quad 0 \le t \le T_0$$

Using Euler Identities:

$$x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \frac{1}{2} e^{j\omega_0 t} + 0 + \frac{1}{2} e^{-j\omega_0 t}$$

$$c[k] = \begin{cases} \frac{1}{2}, & k = 1 \\ \frac{1}{2}, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $$x(t) = \sin(\omega_0 t) = \sin\left(\frac{2\pi}{T_0} t\right)$$
$$0 \le t \le T_0$$

Using Euler Identities:

$$x(t) = \frac{1}{2j}\left\{e^{j\omega_0 t} - e^{-j\omega_0 t}\right\}$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$c[k] = \begin{cases} \frac{1}{2j}, & k = 1 \\ -\frac{1}{2j}, & k = -1 \end{cases}$$

(c) $$x(t) = \cos(2\omega_0 t) + \cos(4\omega_0 t)$$

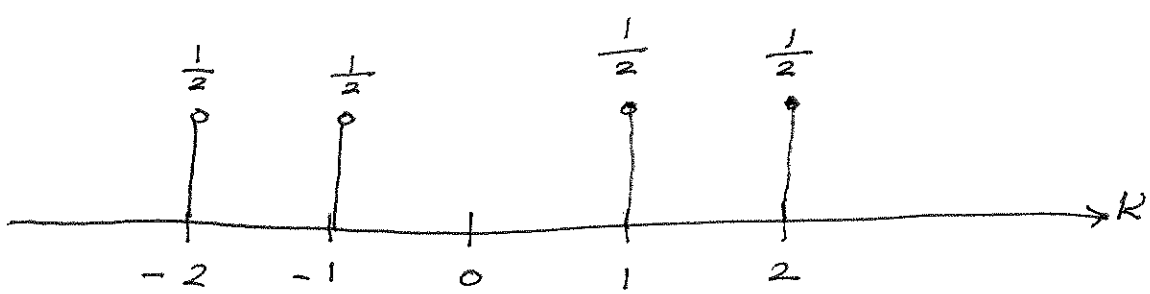
$\cos(2\omega_0 t) \downarrow$ Periodic with $T_1 = \frac{2\pi}{2\omega_0}$

$\cos(4\omega_0 t) \downarrow$ Periodic with $T_2 = \frac{2\pi}{4\omega_0}$

$$T = \mathrm{lcm}\left\{ \frac{2\pi}{2\omega_0}, \frac{2\pi}{4\omega_0} \right\} = \frac{2\pi}{2\omega_0}$$

$$x(t) = \frac{1}{2} e^{j2\omega_0 t} + \frac{1}{2} e^{-j2\omega_0 t} + \frac{1}{2} e^{j4\omega_0 t} + \frac{1}{2} e^{-j4\omega_0 t}$$

$$c[k] = \begin{cases} k = -2, & \frac{1}{2} \\ k = -1, & \frac{1}{2} \\ k = 1, & \frac{1}{2} \\ k = 2, & \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$



(d) $\quad x(t) = \sin(\omega_0 t) - \sin(3\omega_0 t)$

Using Euler Identities:

$$x(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} - \frac{1}{2j} e^{j3\omega_0 t} + \frac{1}{2j} e^{-j3\omega_0 t}$$

$$T_{eff} = \text{lcm}\left\{ \frac{2\pi}{\omega_0}, \frac{2\pi}{3\omega_0} \right\} = \frac{2\pi}{\omega_0}$$

$$c[k] = \begin{cases} \frac{1}{2j}, & k = 1 \\ -\frac{1}{2j}, & k = -1 \\ -\frac{1}{2j}, & k = 3 \\ \frac{1}{2j}, & k = -3 \\ 0, & \text{otherwise} \end{cases}$$