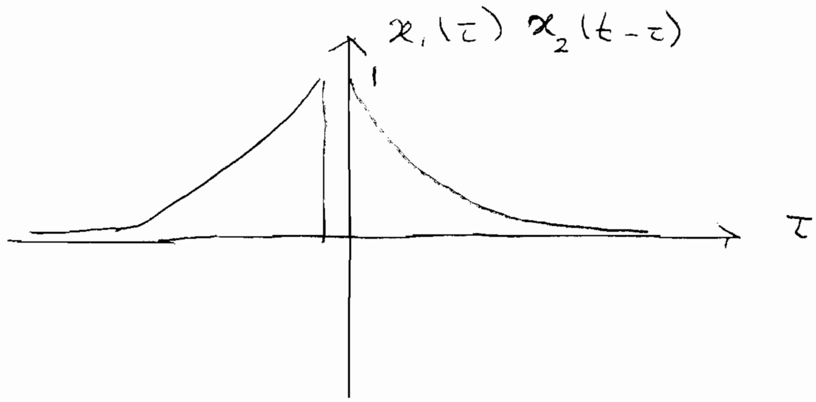
ECE-314, Fall 2008
Signals & Systems

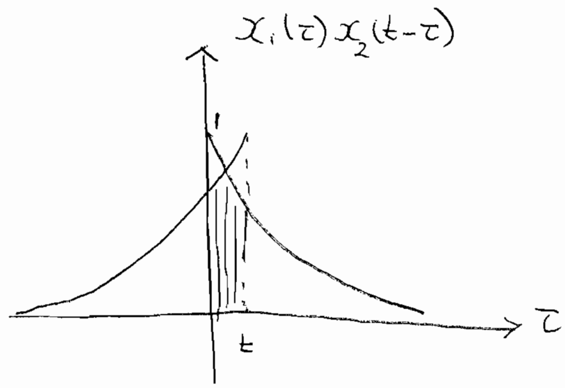
Example: Convolution

$$x_1(t) = e^{-at}\,u(t) \qquad , \quad a \neq b$$
$$x_2(t) = e^{-bt}\,u(t)$$



For $t \le 0$ there is no area under overlap.

$\Rightarrow \quad x_1(t) * x_2(t) = 0$ for $t \le 0$



For $t > 0$:

Area of overlap is in $\tau \in [0, t]$

$$x_1(t) * x_2(t) = \int_0^t e^{-a\tau}\, e^{-b(t-\tau)}\, u(\tau)\, u(t-\tau)\, d\tau$$

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$u(t-\tau) = \begin{cases} 1, & \tau \leq t \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad u(\tau)u(t-\tau) = \begin{cases} 1, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad x_1(t) * x_2(t) = \int_0^t e^{-a\tau} e^{+b\tau} e^{-bt} \, d\tau$$

$$= e^{-bt} \int_0^t e^{+(b-a)\tau} \, d\tau$$

$$= e^{-bt} \left. \frac{e^{(b-a)\tau}}{b-a} \right|_0^t$$

$$= e^{-bt} \left(\frac{e^{(b-a)t}}{b-a} - \frac{1}{b-a} \right)$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}, \quad t \geq 0$$

$$\Rightarrow \quad x_1(t) * x_2(t) = \begin{cases} \dfrac{e^{-at} - e^{-bt}}{b-a}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad x_1(t) * x_2(t) = \left(\frac{e^{-at} - e^{-bt}}{b-a} \right) u(t), \quad b \neq a$$

For $b = a$:

$$x_1(t) * x_2(t) = 0, \quad t \le 0$$

For $t > 0$:

$$x_1(t) * x_2(t) = \int_0^t e^{-a\tau} e^{-a(t-\tau)} \, d\tau$$

$$= e^{-at} \int_0^t d\tau = t e^{-at}$$

$$\Rightarrow x_1(t) * x_2(t) = \begin{cases} t e^{-at}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow x_1(t) * x_2(t) = t e^{-at} u(t)$$

Aside: This result can be generalized as:

$$e^{-at} u(t) * e^{-at} u(t) * e^{-at} u(t) \ldots\ldots * e^{-at} u(t)$$

$$= t^n e^{-at} u(t)$$