LTI Systems and Frequency Response

Uptill now we have used the Dirac impulse function to study the response of a LTI system. The convolution theorem gives us a complete time–domain description of the input-output characteristics of the LTI system.

Let us now look at frequency-domain description of the input-output characteristics of the system by studying the response of the LTI system to a complex exponential:

$$x(t) = \exp\left(j\omega_0 t\right), \quad \omega_o \in \mathbf{R}.$$

The output signal y(t) given by the convolution integral is:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) \exp\left(j\omega_0 (t-\tau)\right) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) \exp\left(-j\omega_0 \tau\right) d\tau \exp\left(j\omega_0 t\right) \\ &= \left[\int_{-\infty}^{\infty} h(\tau) \exp\left(-j\omega_0 \tau\right) d\tau\right] x(t). \end{aligned}$$

Upon defining the *frequency response* of the LTI system $H(\omega)$ as :

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) \exp\left(-j\omega\tau\right) d\tau. \tag{1}$$

we can see that the response of the LTI system to a complex exponential is the same complex exponential multiplied with $H(\omega_o)$, i.e, a complex gain :

$$y(t) = H(\omega_o) x(t) = |H(\omega_o)| \;\exp\left(j \mathrm{Arg}(H(\omega_o))\right) x(t). \tag{2}$$

The quantity $|H(\omega)|$ is called the *magnitude response* and the quantity $\mathrm{Arg}(H(\omega))$ will be refered to as the *phase response* of the LTI system. Note that the frequency response of the LTI system in Eq. (1) is purely a function of the frequency variable: $\omega \in \mathbf{R}$ and is independent of time. The frequency response therefore gives us a picture of what the LTI system does to the input signal interms of the sinusoidal frequency content of the signal.

Alternatively if we were to rewrite this input-output relation in the form:

$$\mathbf{L}(\exp\left(j\omega_o t\right)) = H(\omega_o)(\exp\left(j\omega_o t\right)). \tag{3}$$

This in the form of the eigenvalue equation where $x(t) = \exp\left(j\omega_o t\right)$ is the eigenfunction corresponding to the eigenvalue $H(\omega_o)$. The complex exponential is therefore an eigenfunction of the convolution integral operator $\mathbf{L}$.