2 The Continuous Time Fourier Transform

For signals $x(t)$, that belong to the space of square integrable functions $\mathbf{H}[a,b]$, we define the *Continuous-Time Fourier Transform* (CTFT) pair :

$$\begin{aligned} X(\omega) &= \mathbf{F}\{h(t)\} = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt \\ x(t) &= \mathbf{F}^{-1}\{X(\omega)\} = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t)\, d\omega. \end{aligned}$$

With this definition in mind, let us revisit the complex constant $H(\omega)$ which is the frequency response of the LTI system described by the convolution integral. For a general sinusoidal input: $x(t) = \exp(j\omega t)$, the output of the LTI system given by the convolution integral is

$$y(t) = \mathbf{L}(\exp(j\omega t)) = \underbrace{\left(\int_{-\infty}^{\infty} h(\tau) \exp(-j\omega\tau)\, d\tau\right)}_{H(\omega)} \exp(j\omega t).$$

The complex constant $H(\omega)$, the frequency response of the LTI system, that was derived from the particular solution to the differential equation corresponding to the LTI system excited with a complex exponential input or forcing function from the previous chapter, is, therefore, also the CTFT of the impulse response $h(t)$ of the LTI system:

$$H(\omega) = \mathbf{F}(h(t)) = \int_{-\infty}^{\infty} h(\tau) \exp(-j\omega\tau)\, d\tau.$$

If we are interested in the Fourier coefficient of the periodically extended signal $x_p(t)$, c_n they can be extracted from the CTFT of the the signal as:

$$\begin{aligned} X(n\omega_0) &= \int_{-\infty}^{\infty} x(t) \exp(-jn\omega_0 t)\, dt \\ X(n\omega_0) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \exp(-jn\omega_0 t)\, dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} x_p(t) \exp(-jn\omega_0 t)\, dt = Tc_n \end{aligned}$$

This result is a useful for computing the CTFT of periodic signals indirectly from the Fourier coefficients of the signal $x_p(t)$.