# Example: Bilateral Laplace Inversion

Consider a continuous-time signal whose bilateral Laplace transform is given by the expression:

$$X(s) = \frac{s}{(s+1)(s-1)}.$$

This expression has one zero at $s = 0$ and two poles at $s = \pm 1$. Performing a partial fractions expansion on this expression because it is purely rational:

$$X(s) = \frac{k_1}{s+1} + \frac{k_2}{s-1}.$$

Comparing coefficients or substituting specific values of $s$ we obtain:

$$X(s) = \frac{1/2}{s+1} + \frac{1/2}{s-1}.$$

By definition the ROC of this expression cannot contain singularities. Therefore there are several possibilities for the ROC of this expression:

1. $\sigma < -1$: Region I
2. $\sigma > 1$: Region II
3. $-1 < \sigma < 1$: Region III

Note that of these three possibilities only Region III includes the imaginary axis and therefore will result in a absolutely integrable inverse. Before we begin to investigate the inverse, we will make use of the following Laplace transform pairs:

$$\begin{aligned} \mathcal{L}\left(e^{-at}u(t)\right) &= \frac{1}{s+a},\ \sigma > -a, \\ \mathcal{L}\left(-e^{at}u(-t)\right) &= \frac{1}{s-a},\ \sigma < a. \end{aligned}$$

We will also make use of the pairs obtained by replacing $a$ with $-a$:

$$\begin{aligned} \mathcal{L}\left(e^{at}u(t)\right) &= \frac{1}{s-a},\ \sigma > a, \\ \mathcal{L}\left(-e^{-at}u(-t)\right) &= \frac{1}{s+a}, \sigma < -a. \end{aligned}$$

The solution for the Laplace inverse in this case is given by:

$$x(t) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}, \sigma > -1\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}, \sigma < 1\right).$$

Upon using the appropriate Laplace transform pairs we obtain:

$$x(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{t}u(-t).$$

This solution for the inverse is clearly double-sided and is bounded. In region I, i.e., $\sigma < -1$, we obtain the inverse:

$$x(t) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}, \sigma < -1\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}, \sigma < 1\right)$$

Using the pairs described before:

$$x(t) = -\frac{1}{2}e^{-t}u(-t) - \frac{1}{2}e^{t}u(-t).$$

Clearly this solution produces a non-causal signal and will not be bounded as $t \to -\infty$. For region II, we have the solution:

$$x(t) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}, \sigma > -1\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}, \sigma > 1\right).$$

The corresponding solution for the inverse is:

$$x(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(t).$$

Again this solution is purely causal but will not be bounded as $t \to \infty$.