Chapter 3: Review of the Fourier Transform

The Fourier series representation of signals in the space of square integrable functions assumes periodic extension of the signal. In this chapter we will develop signal processing tools that will enable us to study aperiodic signals also.

1 Limiting Form of the Fourier Series

Consider an aperiodic signal $x(t)$ that has a region of support $\left[-\frac{T}{2}, \frac{T}{2}\right]$. Also consider the periodic signal $x_p(t)$ obtained by periodic extension of the signal $x(t)$:

$$x_p(t) = \sum_{k=-\infty}^{\infty} x(t - kT).$$

The aperiodic signal $x(t)$ is identical to the periodic signal $x_p(t)$ over a the period T, i.e.,

$$\begin{aligned} x(t) &= \lim_{T\to\infty} x_p(t) \\ &= \begin{cases} x_p(t) & \text{if} \quad t \in \left[-\frac{T}{2}, \frac{T}{2}\right] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since the signal $x_p(t)$ is periodic, it has a Fourier series representation of the form:

$$\begin{aligned} x_p(t) &= \sum_{k=-\infty}^{\infty} c_k \exp\left(jk\omega_0 t\right) \\ c_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_p(t) \exp\left(-jk\omega_0 t\right) dt. \end{aligned}$$

As the period T increases, the fundamental frequency of the periodic signal $x_p(t)$, ω_0, which is also the spacing between the harmonics in the Fourier line spectrum of the periodic signal approaches an infinitesimally small quantity $d\omega$, the harmonics $k\omega_0$ approach the continuous variable ω, and the Fourier line spectrum of the periodic signal Tc_k also approaches a continuous spectrum $X(\omega)$:

$$\begin{aligned} \lim_{T\to\infty} c_k T &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_p(t) \exp\left(-j \underbrace{k\omega_0}_{\omega} t\right) dt \\ \lim_{T\to\infty} c_k T &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \exp\left(-j \underbrace{k\omega_0}_{\omega} t\right) dt \\ X(\omega) &= \lim_{T\to\infty} c_k T = \int_{-\infty}^{\infty} x(t) \exp\left(-j\omega t\right) dt \end{aligned}$$

The synthesis relation can also be reformulated in terms of this continuous spectrum $X(\omega)$:

$$\begin{aligned} x_p(t) &= \sum_{k=-\infty}^{\infty} c_k \exp\left(jk\omega_0 t\right) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \left(c_k T\right) \exp\left(jk\omega_0 t\right)\left(\omega_0\right) \\ x(t) &= \lim_{T\to\infty} x_p(t) = \lim_{T\to\infty} \left(\frac{1}{2\pi}\right) \sum_{k=-\infty}^{\infty} \left(c_k T\right) \exp\left(j \underbrace{k\omega_0}_{\omega} t\right) \underbrace{\omega_0}_{d\omega} \\ x(t) &= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} X(\omega) \exp\left(j\omega t\right) d\omega. \end{aligned}$$