Scaling and Reflection

Scaling Operator:

The scaling system described in the class has an input–output map given by

$$y(t) \ = \ \mathbf{S}_a(x(t)) \ = \ x(at), a > 0, \ a \neq 1.$$

Delaying the input signal $x(t)$ by $t_o \in \mathbf{R}$ we obtain the signal $\mathbf{D}_{t_o}(x(t)) = x(t - t_o)$. The response of this scaling system to this delayed input is $\mathbf{S}_a(x(t - t_o))$ which from the mapping of the time axis amounts to:

$$\mathbf{S}_a(\mathbf{D}_{t_o}(x(t))) \ = \ \mathbf{S}_a(x(t - t_o)) \ = \ x(at - t_o).$$

If on the otherhand we interchanged the order of the two systems and applied the scaling operation first we have the relations:

$$\mathbf{D}_{t_o}(\mathbf{S}_a(x(t))) \ = \ \mathbf{D}_{t_o}(x(at)) \ = \ x(at - at_o).$$

These expressions are not the same and therefore the scale operation does not commute with the delay operation. This implies that the scale operation is a *time-varying* (TV) system.

One can gain further insight by looking at the impulse response of the scaling system:

$$h(t, \tau) \ = \ \mathbf{S}_a(\delta(t - \tau)) \ = \ \delta(at - \tau).$$

Using the convolution theorem for the underlying linear system and substituting $\lambda = -at + \tau$ and $d\lambda = d\tau$ we obtain:

$$y(t) \ = \ \int_{-\infty}^{\infty} x(\tau)h(t, \tau)d\tau \ = \ \int_{-\infty}^{\infty} x(\tau)\delta(at - \tau)d\tau \ = \ \int_{-\infty}^{\infty} x(at + \lambda)\delta(\lambda)d\lambda \ = \ x(at).$$

Reflection Operator:

Let us now focus on the reflection operation that has the following input-output map:

$$y(t) = \mathbf{R}(x(t)) = x(-t).$$

The response of the system to a delayed input $x(t - t_o), t_o \in \mathbf{R}$ is given by:

$$\mathbf{R}(\mathbf{D}_{t_o}(x(t))) = \mathbf{R}(x(t - t_o)) = x(-t - t_o).$$

The output of the system delayed by the same amount is given by:

$$\mathbf{D}_{t_o}(\mathbf{R}(x(t))) = \mathbf{D}_{t_o}(x(-t)) = x(-(t - t_o)) = x(-t + t_o).$$

Since these results are not identical the reflection operation and the delay operation do not commute and the system is therefore time-varying.

The corresponding impulse response of the reflection system is given by :

$$h(t, \tau) = R(\delta(t - \tau)) = \delta(-t - \tau).$$