Causality and Stability Revisited

Uptill now we have classified systems as causal or stable by starting from first principles. Let us look at means to identify whether or not a LTI system is causal or stable by looking at its impulse response.

Causality

If we were to excite the system with impulse $h(t)$ with a Dirac impulse, i.e., $x(t) = \delta(t)$ then according to the convolution theorem:

$$y(t) = x(t) * h(t) = h(t) * \delta(t) = h(t).$$

Recall that a causal system is one where the output at the present instant does not anticipate input from future instants. This means in practical terms that

$$x(t), t < t_o \Longleftrightarrow y(t), t < t_o.$$

or in other words "no output before input is applied". The input in this case is the Dirac delta function that is applied at the time instant $t = 0$. This function does not exist before the instant $t = 0$. Therefore for the case with the Dirac delta for the input, a LTI system is causal if and only if :

$$h(t) = \mathbf{L}(\delta(t)) = 0, t < 0.$$

In other words the *impulse response of a LTI system has to be zero for negative time for the system to be causal.*

For a discrete–time system this means that that the impulse response sequence $h[n]$ of a LTI system has to be a *right–sided sequence*, i.e.,

$$h[n] = \mathbf{L}(\delta[n]) = 0, n < 0.$$

Stability

For a LTI system to be *bounded input bounded output* (BIBO) stable, every bounded signal should produce a bounded output. Let us assume that the input signal to this system $x(t)$ is bounded, i.e, $x(t) \leq B, t \in \mathbf{R}$.

Our goal is to find limits or upper-bounds on the output of the system $y(t)$. The convolution theorem is invoked here as a start:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \iff |y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|.$$

Using the triangle inequality we can upper-bound the output $y(t)$ as :

$$y(t) \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|d\tau = \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau = B \underbrace{\int_{-\infty}^{\infty} |h(\tau)|d\tau}_{\text{amplification}}.$$

This quantity has to be finite by definition for the system to be BIBO stable and this inturn implies that the amplification factor is finite, i.e., :

$$|y(t)| < \infty \iff \int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty.$$

Or in other words, *the impulse response has to be absolutely integrable for the LTI system to be BIBO stable.* Similarly the corresponding condition for a discrete–time LTI system to be BIBO stable:

$$\sum_{-\infty}^{\infty} |h[k]| < \infty.$$

Or in other words, *the impulse response has to be absolutely summable for the LTI system to be BIBO stable.*