4 Parsevals Theorem

The goal in this section is to study the effect that the Fourier transform has on the time-domain inner product defined as:

$$< f(t), g(t) > \; = \; \int_{-\infty}^{\infty} f(t) g^*(t) dt.$$

The inverse Fourier Transform relation for the function $g^*(t)$ is obtained as:

$$\begin{aligned} g(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \exp(j\omega t)\, d\omega \\ g^*(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \exp(-j\omega t)\, d\omega. \end{aligned}$$

Substituting this expression into the expression for the inner product expression we have :

$$< f(t), g(t) > \; = \; \int_{-\infty}^{\infty} f(t) \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} G^*(\omega) \exp(-j\omega t)\, d\omega \; dt.$$

Since the integrals are over independent variables and are linear operations, they can be swapped to yield:

$$\begin{aligned} < f(t), g(t) > &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \underbrace{\int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt}_{F(\omega)}\, d\omega \\ < f(t), g(t) > &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) F(\omega) d\omega \; = \; \frac{1}{2\pi} < F(\omega), G(\omega) > . \end{aligned}$$

This relation referred to as the Parsevals theorem in the Fourier domain implies that the inner product relation in the time domain is preserved, except for a scale factor of 2π, in the frequency domain. As a special case, if we were to use $f(t) = g(t)$, then the theorem reduces down to the expression for the energy of the signal E_f as :

$$E_f \; = < f(t), f(t) > = \int_{-\infty}^{\infty} |f(t)|^2 dt \; = \; \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

As an example, if we were to find the energy of the signal: $x(t) \; = \left(\frac{B}{2\pi}\right) \mathrm{Sa}\left(\frac{Bt}{2}\right)$, the expression for the energy via the time-domain innerproduct:

$$E_f \; = \; \left(\frac{B}{2\pi}\right)^2 \int_{-\infty}^{\infty} \mathrm{Sa}^2\left(\frac{Bt}{2}\right) dt$$

is difficult to evaluate. If instead we compute the energy in the frequency domain keeping in mind the Fourier transform pair:

$$x(t) \; = \; \left(\frac{B}{2\pi}\right) \mathrm{Sa}\left(\frac{Bt}{2}\right) \longleftrightarrow X(\omega) \; = \; \mathrm{rect}\left(\frac{\omega}{B}\right),$$

we obtain the equivalent quantity in the frequency-domain easily as:

$$E_f \; = \; \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\mathrm{rect}\left(\frac{\omega}{B}\right)\right]^2 d\omega \; = \; \frac{1}{2\pi} \int_{-\frac{B}{2}}^{\frac{B}{2}} d\omega \; = \; \frac{B}{2\pi}.$$