Parseval's Theorem

Given that we have a function $f(t)$ that belongs to the class of functions $\mathbf{H}[a, b]$ we can expand the function in terms of a basis of orthogonal functions $\{\phi_k(t)\ ,\ -\infty \leq k \leq \infty\}$ as

$$f(t) \ = \ \sum_{k=-\infty}^{\infty} c_k \phi_k(t)$$

If the signal $f(t)$ is an energy signal then its energy E_f is a finite quantity.

$$E_f \ = < f(t), f(t) > = \ \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Substituting the expansion of the function $f(t)$ into the expression for the energy E_f:

$$\begin{aligned}
E_f &= \ < \sum_{m=-\infty}^{\infty} c_m \phi_m(t), \sum_{n=-\infty}^{\infty} c_n \phi_n(t) > \\
E_f &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m c_n^* < \phi_m(t), \phi_n(t) > \\
E_f &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m c_n^* < \phi_n(t), \phi_n(t) > \delta_{m,n} \\
E_f &= \sum_{n=-\infty}^{\infty} |c_n|^2 < \phi_n(t), \phi_n(t) > \ , \quad (\text{Parseval's Theorem})
\end{aligned}$$

where, again, we have interchanged the order of the summation and integration.

For periodic signals, we use the basis of complex exponentials: $\{\phi_k(t) \ = \ \exp(jk\omega_0 t)\}$. In this case of a periodic signal $f(t)$, the energy, E_f, of the periodic signal becomes infinite and it is then more appropriate to talk of the average power over a single period:

$$\mathrm{P_{ave}} \ = \ \frac{1}{T_0} \int_0^{T_0} f(\tau) f^*(\tau) d\tau \ = \ \frac{1}{T_0} \int_0^{T_0} |f(t)|^2 dt.$$

Following the same procedure as in the case of the energy signals we can substitute the Fourier series expansion for $f(t)$ in the expression for the average power and obtain:

$$\begin{aligned}
\mathrm{P_{ave}} &= \frac{1}{T_0} \int_0^{T_0} f(\tau) f^*(\tau) d\tau \\
\mathrm{P_{ave}} &= \frac{1}{T_0} \int_0^{T_0} \sum_{m=-\infty}^{\infty} c_m \phi_m(\tau) \sum_{n=-\infty}^{\infty} c_n^* \phi_n^*(\tau) d\tau \\
\mathrm{P_{ave}} &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m c_n^* \frac{1}{T_0} \underbrace{\int_0^{T_0} \phi_m(\tau) \phi_n^*(\tau) d\tau}_{<\phi_m(t), \phi_n(t)> \ = \ T_0 \delta_{m,n}} \\
\mathrm{P_{ave}} &= \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad (\text{Parseval's Theorem})\,.
\end{aligned}$$

This is a very useful relation for computing the power because it says that the average power $\mathrm{P_{ave}}$ of the periodic signal $f(t)$ can be computed both from the signal directly or indirectly through the Fourier coefficients of the signal.