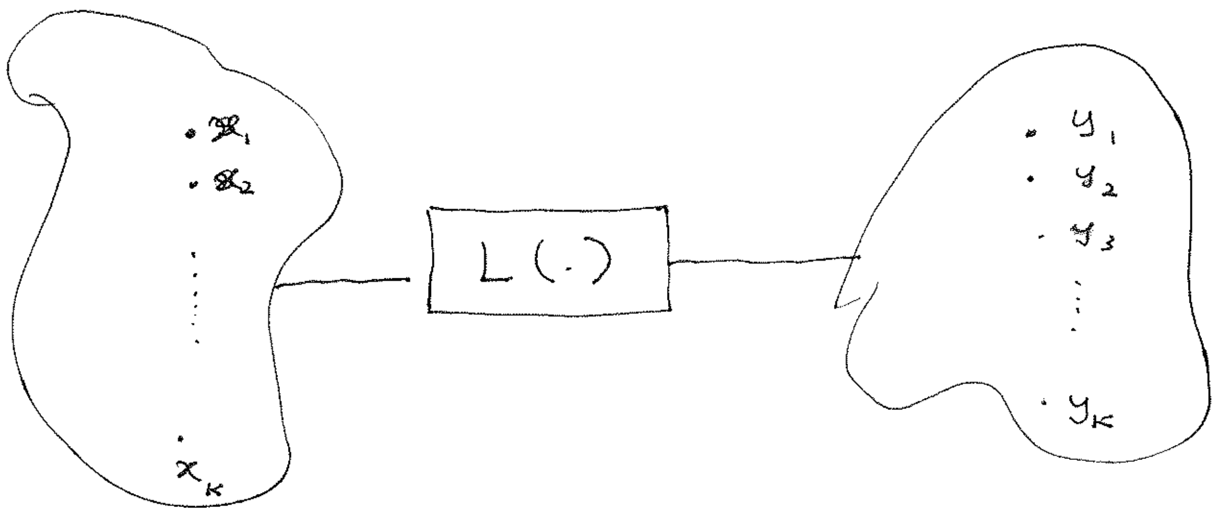
ECE - 314, Fall 2018
Signals & Systems

Linear Systems & Superposition



Input Space　　　　　　　　Output space

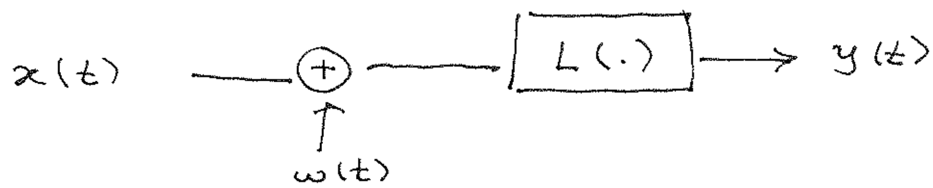
A system map $L(.)$ is a transformation from the space of input signals to the space of output signals

The system map $L(.)$ is said to be linear if principle of superposition applies, i.e.,

$$L(c_1 x_1 + c_2 x_2) = c_1 L(x_1) + c_2 L(x_2)$$

MOTIVATION

Linearity is specifically useful in systems where additive noise is present



$$y(t) = L(x(t) + w(t))$$
$$= L(x(t)) + L(w(t))$$

In otherwords the overall response is just the addition of the response to both the signal portion and the noise portion.

Example :

$$y(t) = L(x(t)) = D_{t_0}(x(t)) = x(t-t_0)$$

$$L(c_1 x_1(t) + c_2 x_2(t)) = D_{t_0}(c_1 x_1(t) + c_2 x_2(t))$$
$$= c_1 x_1(t-t_0) + c_2 x_2(t-t_0)$$
$$= c_1 L(x_1(t)) + c_2 L(x_2(t))$$

$\Rightarrow L(\cdot)$ is a linear system

Example:

$$y(t) = L(x(t)) = a\,x(t) + b$$

$$L(c_1 x_1(t) + c_2 x_2(t)) = a\,c_1 x_1(t) + a\,c_2 x_2(t) + b \quad (\mathrm{I})$$

$$L(x_1(t)) = a\,x_1(t) + b$$
$$L(x_2(t)) = a\,x_2(t) + b$$

$$c_1 L(x_1(t)) + c_2 L(x_2(t))$$

$$= a\,c_1 x_1(t) + c_1 b + a\,c_2 x_2(t) + c_2 b$$

$$= a\,(c_1 x_1(t) + c_2 x_2(t)) + b(c_1 + c_2) \quad (\mathrm{II})$$

$$\mathrm{I} \neq \mathrm{II}$$

$\Rightarrow$ $L(.)$ is not a linear map.

$\Rightarrow$ $L.()$ is a affine map

Example - II

$$y[n] = a x[n-1] + x[n-2]$$
$$= L(x[n])$$

$$L(c_1 x_1[n] + c_2 x_2[n])$$

$$= a(c_1 x_1[n-1] + c_2 x_2[n-1])$$
$$+ c_1 x_1[n-2] + c_2 x_2[n-2]$$

$$= c_1\{a x_1[n-1] + x_1[n-2]\}$$
$$+ c_2\{a x_2[n-1] + x_2[n-2]\}$$

$$= c_1 L(x_1[n]) + c_2 L(x_2[n])$$

$\Longrightarrow$ POS applies and $L(\cdot)$ is a linear system

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Time / Shift Invariance

Denoting the operation of shifting a signal by t_0 sec as $D_{t_0}(\cdot)$:

$$x(t-t_0) = D_{t_0}(x(t))$$

A system operator $L(\cdot)$ is a time-invariant operation :

$$D_{t_0}(L(x(t))) = L\left(D_{t_0}(x(t))\right)$$

or

$$y(t-t_0) = L(x(t-t_0))$$

$\Rightarrow$ Response to a delayed input is a delayed output

$\Rightarrow$ Important from measurement origin perspective

$\Rightarrow$ $D_{t_0} L(x(t)) = L D_{t_0}(x(t))$

$\Rightarrow$ $[L, D_{t_0}] = \underline{0}$

Example:

$$y(t) = \frac{d}{dt}(x(t)) = L(x(t))$$

$$= \lim_{\Delta t \to 0} \frac{x(t) - x(t-\Delta t)}{\Delta t}$$

$$L(x(t-t_0)) = \lim_{\Delta t \to 0} \frac{x(t-t_0) - x(t-t_0-\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{x(t) - x(t-\Delta t)}{\Delta t} \Bigg|_{t \to t-t_0}$$

$$= y(t-t_0)$$

$\Rightarrow$ System is time-invariant

Example - II

$$y[n] = x[3n+2]$$
$$= L(x[n])$$

$$L(x[n-n_0]) = x[3n+2-n_0]$$

$$y[n-n_0] = x[3(n-n_0)+2]$$
$$= x[3n-3n_0+2]$$

Since $y[n-n_0] \neq L(x[n-n_0])$
$L(.)$ is time-varying.

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BIBO Stability

A system map $y(t) = L(x(t))$ is said to be BIBO stable if

for any $|x(t)| < B_{in} < \infty$ the corresponding $|y(t)| < B_{out} < \infty$

(Bounded - Input Bounded - output stable)

→ Physical systems are BIBO stable

→ Unstable systems are undesirable (Collapsing of a bridge under duress)

Example - I

$$y(t) = t\,x(t) = L_1(x(t))$$

Suppose $|x(t)| < B_1 < \infty$

$|y(t)|$ cannot be bounded by a box for any $t \in \mathbb{R}^1$

$\Rightarrow L_1(\cdot)$ is not BIBO stable

Example – II

$$y(t) = L_2(x(t)) = \frac{1}{x(t) - 1}$$

Suppose $x(t) = 1$, $t \in \mathbb{R}$

$|x(t)| \leq 1 = B_1 < \infty$ (Bounded Input)

$|y(t)|$ for this $x(t) = \infty$

(Unbounded output)

Example – III

$$y(t) = L_3(x(t)) = |x(t)|$$

For a bounded Input : $|x(t)| < B_1 < \infty$

the output is always bounded:

$$|y(t)| = |x(t)| < B_1 < \infty$$

$\Rightarrow$ $L_3(\cdot)$ is a BIBO system

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Causality:

$y(t) = L(x(t))$ is said to be a causal system if for any $t \in \mathbb{R}$ $y(t)$ does not anticipate future input

Similarly

$y[n] = L(x[n])$ is said to be causal if present output depends on future input

Importance

→ Memory or storage needed for non-causal systems

→ Typically cache-memory

→ Online-computations require causality

→ ~~fMRI/MRI data for single~~ patient requires DVD worth of storage needs off-line computation

Example:

$$y(t) = 3x(t-2) + 1$$

For $t \in R$ $\quad t > t-2$

$\Rightarrow$ present output only depends on past & does not anticipate

$\Rightarrow$ $L(\cdot)$ is a causal system map

Example - II

$$y[n] = x[3n] = L(x[n])$$

For $n \geq 0$ $\quad 3n \geq n$

$\Rightarrow$ output anticipatory for $n \geq 0$

For $n < 0$ $\quad 3n < n$

$\Rightarrow$ non - anticipatory for $n < 0$

$\Rightarrow$ Overall system is an anticipatory system

$\Rightarrow$ $L(\cdot)$ is a non - causal system map

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INVERTIBILITY

A system map $y[n] = L(x[n])$ is said to be an invertible system if the effect of the system can be reversed, i.e.,

$$\exists\ L^{-1}(\cdot) \ni L^{-1}(y[n]) = x[n]$$

or

$$L^{-1}(L(x[n])) = x[n]$$

MOTIVATION

(a) In many applications $L(\cdot)$ represents channel distortion or blurring in images

(b) $L^{-1}(\cdot)$ operation would correspond to channel equalization or image deblurring

(c) Issue: $L^{-1}(\cdot)$ may not exist, i.e, may be unbounded or unrealizable

Example - I

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = L(x(t))$$

Applying the derivative operation to $y(t)$:

$$\frac{dy}{dt} = \frac{d}{dt}\left\{ \int_{-\infty}^{t} x(\tau)\, d\tau \right\}$$

By fundamental theorem of calculus

$$\frac{dy}{dt} = x(t)$$

$$\Rightarrow \frac{d}{dt}\left(L(x(t))\right) = x(t)$$

$\Rightarrow \frac{d}{dt}(\cdot)$ is the corresponding L^{-1}

$\Rightarrow L(\cdot)$ is inventible

Example

$$y[n] = x^2[n] = L(x[n])$$

Denoting the square-root operation by $(\cdot)^{1/2}$

$$\sqrt{y[n]} = (y[n])^{1/2} = |x[n]|$$

$\Rightarrow$ Cannot recover $x[n]$

$\Rightarrow L(\cdot)$ not inventible.