3 Fourier Transform of Periodic Signals

In this section we look at the Fourier transform relation for periodic signals. We know from chapter 2 that every periodic signal $f(t)$ can be expressed in terms of its Fourier series as:

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} c_k \exp(jk\omega_0 t) \\ c_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_p(t)\exp(-jk\omega_0 t)\,dt. \end{aligned}$$

Our goal in particular is to find the inverse Fourier transform of the function:

$$X(\omega) = \sum_{k=-\infty}^{\infty} (2\pi c_k)\,\delta(w - k\omega_0)$$

Substituting the above expression in to the inverse Fourier transform relation we have:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\,d\omega \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}(2\pi c_k)\,\delta(w-k\omega_0)\exp(j\omega t)\,d\omega. \end{aligned}$$

Since the summation and the integral variables are independent and both the sum and the integral are linear operations they can be interchanged to yield:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty}\exp(j\omega t)\,\delta(w-k\omega_0)d\omega.$$

The sampling property of the delta function can then be evoked to obtain the expressions:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty}\exp(jk\omega_0 t)\,\delta(w-k\omega_0)d\omega \\ x(t) &= \sum_{k=-\infty}^{\infty} c_k \exp(jk\omega_0 t)\underbrace{\int_{-\infty}^{\infty}\delta(w-k\omega_0)d\omega}_{1} \end{aligned}$$

This yields the Fourier transform pair for periodic signals:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\exp(jk\omega_0 t) \quad\longleftrightarrow\quad X(\omega) = \sum_{k=-\infty}^{\infty}(2\pi c_k)\,\delta(w-k\omega_0).$$

The spectrum of a periodic signal is therefore a discrete spectrum with impulses at the frequencies: $\omega = k\omega_0, -\infty \le k \le \infty$ with corresponding areas of $2\pi c_k$.