ECE -314, Fall 2008
Signals and Systems

(1) Review Session II

A discrete-time LTI system has an impulse response given by:

$$h[n] = s^{|n|}, \quad 0 < s < 1, \; s \in \mathbb{R}$$

DTFT

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\Omega n}$$

$h[n]$ in our case is given by:

$$\begin{cases} s^{n}, & n \geq 0 \\ s^{-n}, & n \leq 1 \end{cases}$$

or

$$h[n] = s^{n} u[n] + s^{-n} u[-n-1]$$

$$H(e^{j\Omega}) = \sum_{n=0}^{\infty} s^{n} e^{-j\Omega n} + \sum_{n=-\infty}^{-1} s^{-n} u[-n-1]\, e^{-j\Omega n}$$

$$H(e^{j\Omega}) = \frac{1}{1-\rho e^{-j\Omega}} + \sum_{n=-\infty}^{-1} \rho^{-n} e^{-j\Omega n}$$

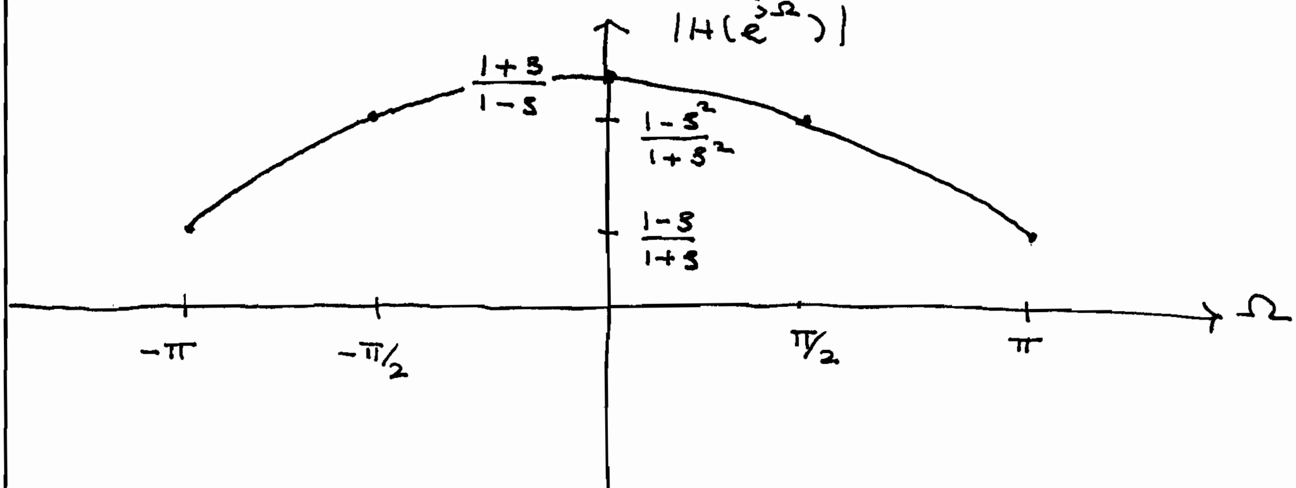
$$H(e^{j\Omega}) = \frac{1}{1-\rho e^{-j\Omega}} + \sum_{m=\infty}^{1} \left(\rho e^{j\Omega}\right)^{m}$$

$$H(e^{j\Omega}) = \frac{1}{1-\rho e^{-j\Omega}} + \left(\frac{1}{1-\rho e^{j\Omega}} - 1\right)$$

$$H(e^{j\Omega}) = \frac{1}{1-\rho e^{-j\Omega}} + \frac{\rho e^{j\Omega}}{1-\rho e^{j\Omega}}$$

$$H(e^{j\Omega}) = \frac{1-\rho e^{j\Omega} + \rho e^{j\Omega}\left(1-\rho e^{-j\Omega}\right)}{1+\rho^2-2\rho\cos\Omega}$$

$$H(e^{j\Omega}) = \frac{1-\rho^2}{1+\rho^2-2\rho\cos\Omega}, \quad |\Omega| < \pi$$



Observations :

(a) $h[n]$ is obviously a non-causal system since, $h[n] \neq 0, \; n < 0$

(b) $h[n]$ corresponds to a stable system since :

$$|H(e^{j\Omega})| < \infty, \; \forall \Omega \in [-\pi, \pi]$$

Difference Eq :

$$\frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1 - s^2}{1 + s^2 - s(e^{j\Omega} + e^{-j\Omega})}$$

$$(1 + s^2)\, y[n] - s\, y[n+1] - s\, y[n-1] = (1 - s^2)\, x[n]$$

$$-s\, \underbrace{y[n+1]}_{\text{Non-causal term}} + (1 + y^2)\, y[n] - s\, y[n-1] = (1 - s^2)\, x[n]$$

Problem 2.0:

(a) $x(t) = \Pi_T(t)\cos(\omega_0 t)$, $\Pi_T(t) = \begin{cases} 1, & |t| < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$

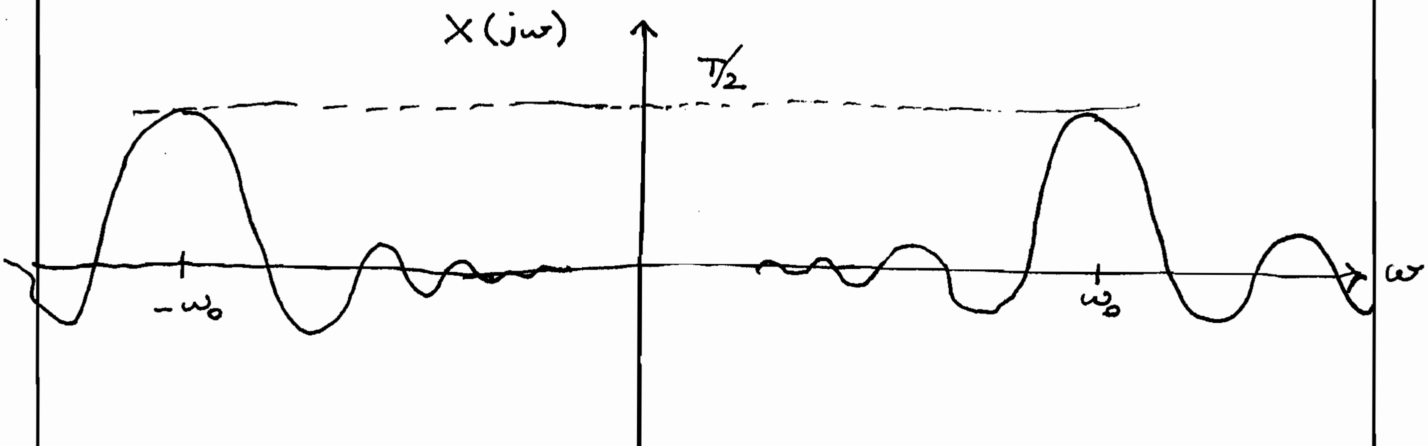
(a) $X(j\omega) = \frac{1}{2\pi}\{\mathcal{F}(\Pi_T(t)) * \mathcal{F}(\cos(\omega_0 t))\}$

$\mathcal{F}\{\cos(\omega_0 t)\} = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$

$X(j\omega) = \frac{1}{2}P(j(\omega+\omega_0)) + \frac{1}{2}P(j(\omega-\omega_0))$,

where $P(j\omega) = \mathcal{F}\{\Pi_T(t)\} = T\,Sa\left(\frac{\omega T}{2}\right)$

$X(j\omega) = \frac{T}{2}\,Sa\left(\frac{(\omega+\omega_0)T}{2}\right) + \frac{T}{2}\,Sa\left(\frac{(\omega-\omega_0)T}{2}\right)$



Here, the underlying assumption:

$\omega_0 \gg \frac{4\pi}{T}$ (For commercial FM the ratio of carrier freq to message BW is very large)

(b)

$$x(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$x(t) = e^{-t}\left(u(t) - u(t-1)\right)$$

$$x(t) = e^{-t}u(t) - \frac{1}{e}\, e^{-(t-1)}u(t-1)$$

$$X(j\omega) = \mathcal{F}\{e^{-t}u(t)\} - \frac{1}{e}\mathcal{F}\{e^{-(t-1)}u(t-1)\}$$

$$X(j\omega) = \frac{1}{1+j\omega} - \frac{1}{e}\mathcal{F}\{e^{-(t-1)}u(t-1)\}$$

$$\mathcal{F}\{e^{-(t-1)}u(t-1)\} = \frac{1}{1+j\omega}\, e^{-j\omega(1)}$$

$$X(j\omega) = \frac{1}{1+j\omega} - \frac{1}{e}\,\frac{1}{1+j\omega}\, e^{-j\omega}$$

$$X(j\omega) = \frac{1}{1+j\omega} - \frac{1}{1+j\omega}\, e^{-(1+j\omega)}$$

$$X(j\omega) = \frac{1}{1+j\omega}\left(1 - e^{-(1+j\omega)}\right)$$

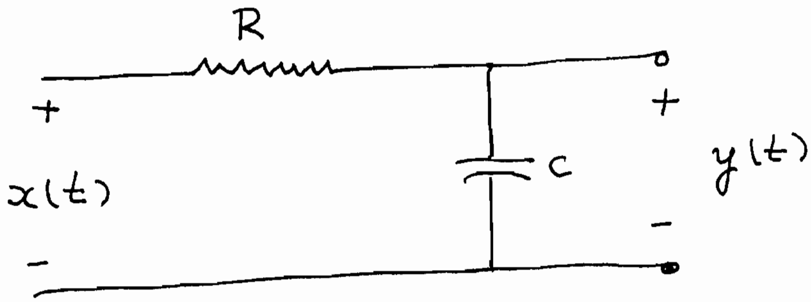
Direct Evaluation:

$$X(j\omega) = \int_0^1 e^{-t}e^{-j\omega t}\,dt = \frac{1}{-(1+j\omega)}\, e^{-(1+j\omega)t}\Big|_0^1$$

$$= \frac{-1}{1+j\omega}\left\{e^{-(1+j\omega)} - 1\right\}$$

$$= \frac{1 - e^{-(1+j\omega)}}{1+j\omega}$$

Problem # 3.0



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$b_k = a_k H(k\omega_0), \quad \text{where}$$

$$H(j\omega) = \frac{a}{a+j\omega}, \quad a = \frac{1}{RC}$$

$$H(k\omega_0) = \frac{a}{a+jk\omega_0}$$

$$b_k = a_k \cdot \frac{a}{a+jk\omega_0}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \frac{1/RC}{1/RC + jk\omega_0} e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \frac{1}{1+jk\omega_0 RC} e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \frac{1}{\sqrt{1+k^2\omega_0^2 R^2 C^2}} e^{j(k\omega_0 t - \tan^{-1}(k\omega_0 RC))}$$

Suppose $\omega_0 >>>> \frac{1}{RC}$

$$|H(k\omega_0)| = \frac{a}{\sqrt{a^2 + k^2\omega_0^2}}$$

$$\approx \frac{a}{k\omega_0} \to 0 \quad \text{as} \quad k \to \infty$$

For $\omega_0 << \frac{1}{RC}$

$$|H(k\omega_0)| = \frac{1}{\sqrt{1 + \frac{k^2\omega_0^2}{a^2}}} \approx 1$$

$\Rightarrow$ Harmonics much larger than $\frac{1}{RC}$ will be significantly attenuated

$\Rightarrow$ Harmonics less than $\frac{1}{RC}$ will experience relatively less distortion and pass through un-attenuated.

Suppose $x(t)$ is a impulse train

$$x(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T_s}t}, \quad a_k = \frac{1}{T_s}, \ \forall k$$

$$y(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} \frac{1}{\sqrt{1 + \frac{k^2 4\pi^2 R^2 C^2}{T_s^2}}} e^{j\left(k\frac{2\pi}{T_s}t - \tan^{-1}\left(k\frac{2\pi RC}{T_s}\right)\right)}$$