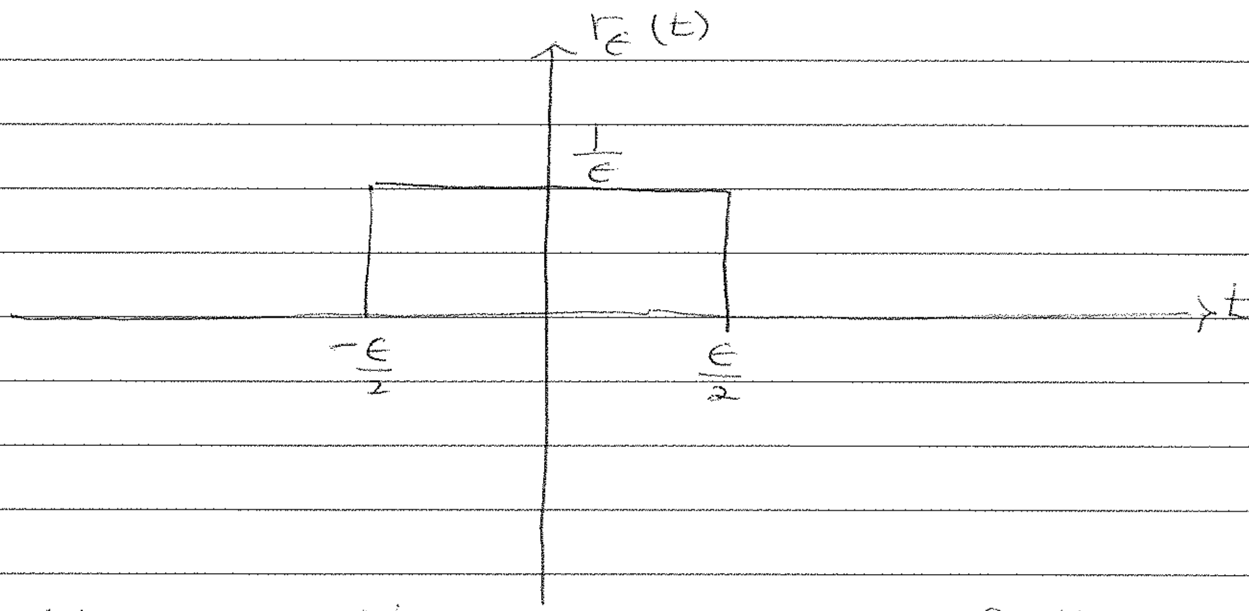
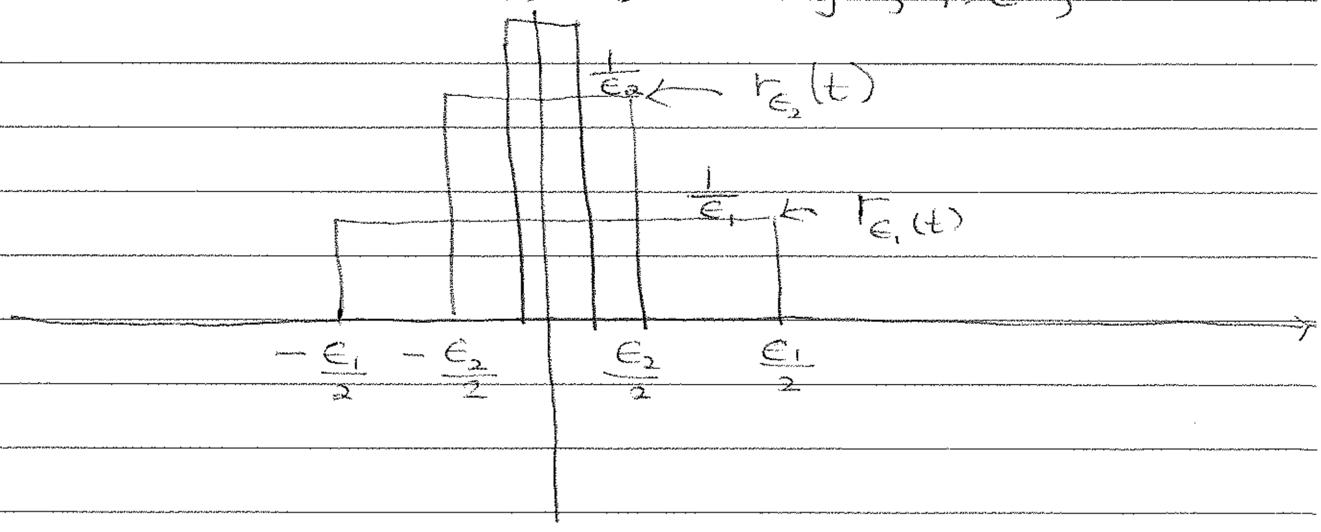
ECE-314, Fall 2018
Signals & Systems

Singularity Functions

Consider the rectangular pulse $r_\epsilon(t)$ with pulse width ϵ and unit area:



Now consider a sequence of these rectangular functions with unit area whose base is shrinking, i.e.,



The Dirac "Delta function" denoted $\delta(t)$ is the limit of the sequence of functions obtained in the limit as the width shrinks to zero:

$$\underset{\text{DIRAC}}{\delta(t)} = \lim_{\epsilon \to 0} r_\epsilon(t)$$

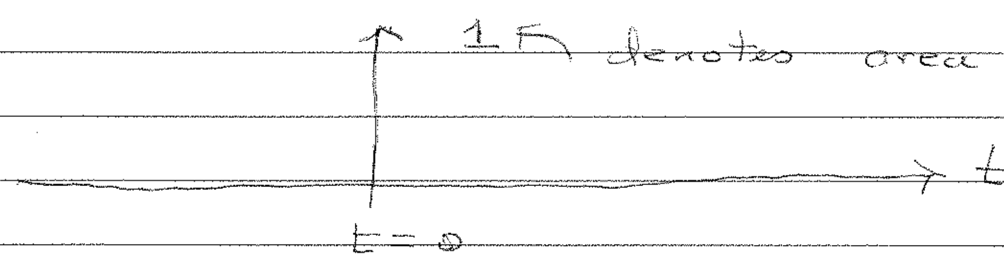
The fundamental characteristic of $\delta(t)$ is that it has unit area

$$\text{Area}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad (a)$$

and that it is zero for $t \neq 0$:

$$\delta(t) = 0, \quad t \neq 0 \qquad (b)$$

Notation for $\delta(t)$:



Other properties of $\delta(t)$:

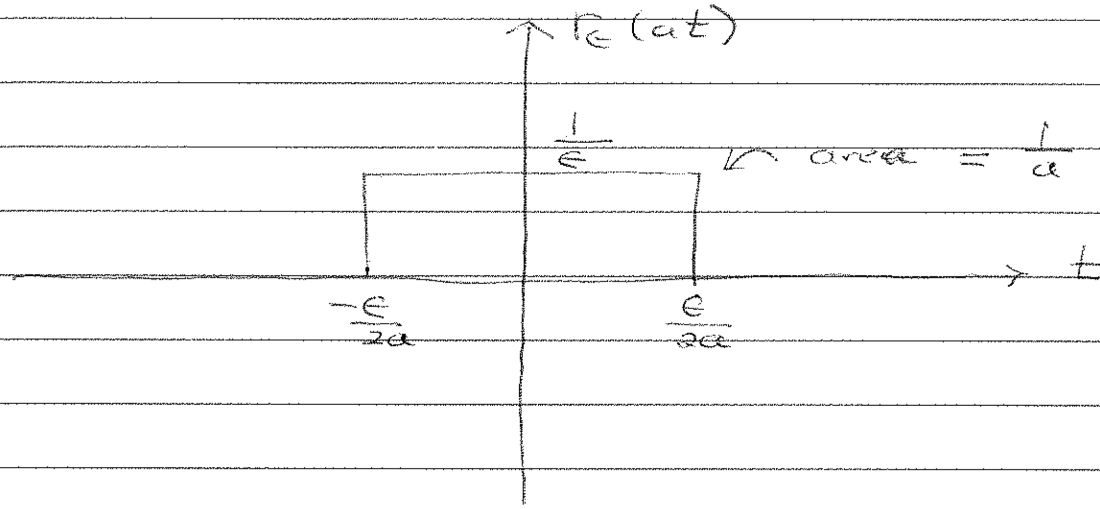
(a) $\delta(-t) = \lim_{\epsilon \to 0} r_\epsilon(-t) = \lim_{\epsilon \to 0} r_\epsilon(t)$

$= \delta(t)$

This implies that $\delta(t)$ is an even "generalized function"

(2) $$\delta(at) \triangleq \lim_{\epsilon \to 0} r_\epsilon(at)$$

whose area is $\frac{1}{a}$ as depicted by



Alternatively:

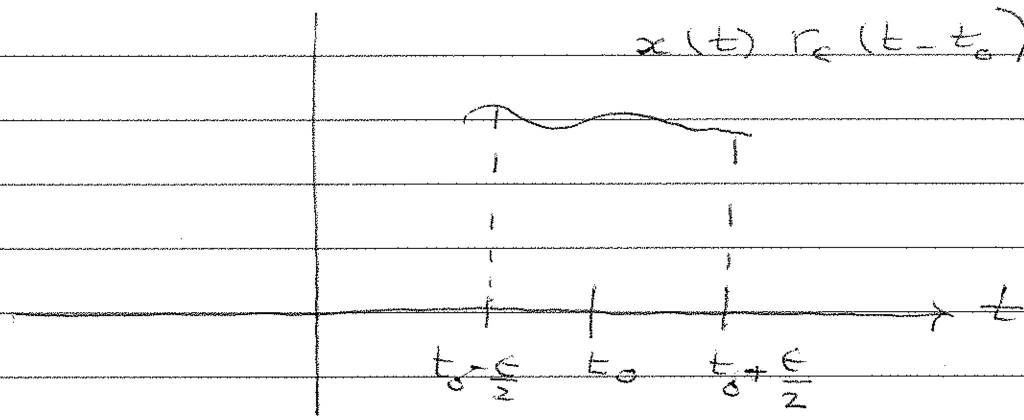
$$\int_{-\infty}^{\infty} \delta(at)\,dt = \int_{-\infty}^{\infty} \delta(\tau)\,\frac{d\tau}{|a|}$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(\tau)\,d\tau = \frac{1}{|a|}$$

$\Rightarrow$ $\delta(at)$ has the same effect as $\frac{1}{|a|}\delta(t)$

(c) Consider the product:

$x(t)\, r_\epsilon(t-t_0)$, where $r_\epsilon(t)$ is the rectangular pulse with unit area and width ϵ



ϵ can be made arbitrarily small so that if $x(t)$ is continuous:

$$x(t)\, r_\epsilon(t-t_0) \simeq x(t_0)\, r_\epsilon(t-t_0)$$

Proceeding to limits

$$\lim_{\epsilon \to 0} x(t)\, r_\epsilon(t-t_0) = \lim_{\epsilon \to 0} x(t_0)\, r_\epsilon(t-t_0)$$

or

$$x(t)\, \delta(t-t_0) = x(t_0)\, \delta(t-t_0)$$

This is the sifting property of the Dirac impulse function

A direct consequence of the sifting property is :

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt$$

$$= x(t_0)\overbrace{\int_{-\infty}^{\infty}\delta(t-t_0)\,dt}^{1}$$

$$= x(t_0)$$

Rewriting this relation in general

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

This result is called the sifting theorem

(d) $$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \text{Area under } (-\infty, t] \text{ of } \delta(t)$$

$$= \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= u(t) \quad \left(\begin{array}{c}\text{Heavy-side or} \\ \text{unit step function}\end{array}\right)$$

Another way of representing a Dirac impulse is :

$$\delta(t) = \frac{du(t)}{dt}$$

(e) $$\delta^{(1)}(t) \triangleq \frac{d\delta(t)}{dt}$$

$$= \lim_{h \to 0} \frac{\delta(t+\frac{h}{2}) - \delta(t-\frac{h}{2})}{h}$$

(i) $$= \lim_{h \to 0} \frac{1}{h}\delta(t+\tfrac{h}{2}) - \lim_{h \to 0} \frac{1}{h}\delta(t-\tfrac{h}{2})$$

Dirac Impulse at $t = \frac{-h}{2}$ with area $\frac{1}{h}$

Dirac impulse at $t = \frac{h}{2}$ with area $\frac{1}{h}$

(ii) $$\int_{-\infty}^{\infty} \delta^{(1)}(t)\,dt = \lim_{h \to 0} \frac{1}{h} \overbrace{\int_{-\infty}^{\infty} \delta(t+\tfrac{h}{2})\,dt}^{1}$$

$$- \lim_{h \to 0} \frac{1}{h} \overbrace{\int_{-\infty}^{\infty} \delta(t-\tfrac{h}{2})\,dt}^{1}$$

$$= 0$$

(iii) $$\delta^{(1)}(-t) = \lim_{h \to 0} \frac{1}{h}\delta(t-\tfrac{h}{2}) - \lim_{h \to 0} \frac{1}{h}\delta(t+\tfrac{h}{2})$$

$$= -\delta^{(1)}(t) \quad \text{(Odd function)}$$

$(\delta(-t+\frac{h}{2}) = \delta(t-\frac{h}{2}))$

(iv) $$\int_{-\infty}^{\infty} x(t)\, \delta^{(1)}(t - t_0)\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\lim_{h \to 0} \frac{1}{h} \delta\left(t - t_0 + \frac{h}{2}\right) - \frac{1}{h} \delta\left(t - t_0 - \frac{h}{2}\right) \right] dt$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{-\infty}^{\infty} x(t)\, \delta\left(t - t_0 + \frac{h}{2}\right) dt$$

$$- \lim_{h \to 0} \frac{1}{h} \int_{-\infty}^{\infty} x(t)\, \delta\left(t - t_0 - \frac{h}{2}\right) dt$$

Using Sifting Theorem:

$$I = \lim_{h \to 0} \frac{1}{h} x\left(t_0 - \frac{h}{2}\right) - \lim_{h \to 0} \frac{1}{h} x\left(t_0 + \frac{h}{2}\right) = - x^{(1)}(t_0)$$

Generalizing the result

$$\int_{-\infty}^{\infty} x(t)\, \delta^{(n)}(t - t_0)\, dt = (-1)^n x^{(n)}(t_0)$$

NOTE: $\delta(t)$ and $\delta^{(n)}(t)$ are idealizations. They are not physical signals and not realizable