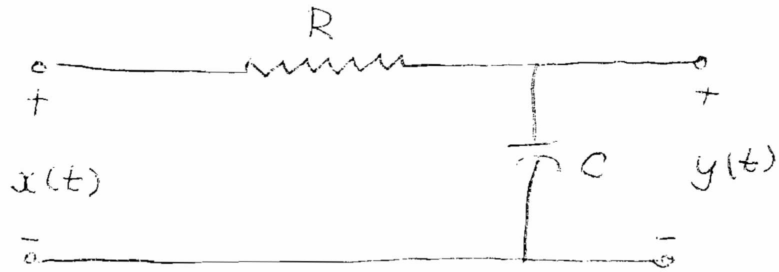
ECE - 314, Fall 2008
Signals & Systems

Sinusoidal Response of R-C circuit :



$$x(t) = RC\frac{dy}{dt} + y(t)$$

Let us look at $y(t)$ for $x(t) = e^{j\omega_0 t}$

$$e^{j\omega_0 t} = RC\frac{dy}{dt} + y(t)$$

Specifically $y(t) = H(j\omega_0)\,e^{j\omega_0 t}$, where

$$H(j\omega_0) = \int_{-\infty}^{\infty} h(t)\,e^{-j\omega_0 t}\,dt$$

$$\Rightarrow \cancel{e^{j\omega_0 t}} = RC\,j\omega_0\,H(j\omega_0)\,e^{j\omega_0 t} + H(j\omega_0)\,\cancel{e^{j\omega_0 t}}$$

$$\Rightarrow H(j\omega_0) = \frac{1}{RCj\omega_0 + 1} = \frac{1/RC}{j\omega_0 + 1/RC}$$

Similarly $\quad L(e^{-j\omega_0 t}) = H(-j\omega_0)\, e^{-j\omega_0 t}$

$$L(e^{+j\omega_0 t}) = H(j\omega_0)\, e^{+j\omega_0 t}$$

$$L(e^{-j\omega_0 t}) = H(-j\omega_0)\, e^{-j\omega_0 t}$$

$$L\left(\tfrac{1}{2} e^{j\omega_0 t} + \tfrac{1}{2} e^{-j\omega_0 t}\right) = \tfrac{1}{2} H(j\omega_0)\, e^{j\omega_0 t} + \tfrac{1}{2} H(-j\omega_0)\, e^{-j\omega_0 t}$$

$$L\left(\cos(\omega_0 t)\right) = \frac{1}{2}\, \frac{a}{a+j\omega_0}\, e^{j\omega_0 t} + \frac{1}{2}\, \frac{a}{a-j\omega_0}\, e^{-j\omega_0 t}$$

$$, \quad a = \frac{1}{RC}$$

$$L\left(\cos(\omega_0 t)\right) = \frac{a}{2}\, \frac{1}{\sqrt{a^2+\omega_0^2}}\, e^{-j\tan^{-1}\left(\frac{\omega_0}{a}\right)}\, e^{j\omega_0 t} + \frac{a}{2}\, \frac{1}{\sqrt{a^2+\omega_0^2}}\, e^{+j\tan^{-1}\left(\frac{\omega_0}{a}\right)}\, e^{-j\omega_0 t}$$

$$L\left(\cos(\omega_0 t)\right) = \frac{a}{2\sqrt{a^2+\omega_0^2}} \left(e^{j(\omega_0 t - \phi)} + e^{-j(\omega_0 t - \phi)} \right)$$

$$= \frac{a}{\cancel{2}\sqrt{a^2+\omega_0^2}}\, \cancel{2} \cos(\omega_0 t - \phi)$$

$$= \frac{a}{\sqrt{a^2+\omega_0^2}} \cos\left(\omega_0 t - \tan^{-1}\left(\frac{\omega_0}{a}\right)\right)$$

Observations:

The output of the R-C circuit to a cosinusoidal input has 2 components

- Magnitude scaling by $|H(j\omega)|$
- Phase shift by $\mathrm{Arg}\,(H(j\omega))$

- $|H(j\omega)|$ is sometimes called the magnitude response
- $\mathrm{Arg}\{H(j\omega)\}$ is called the phase response
- $H(j\omega)$ is called the frequency response of the R-C circuit