# Trigonometric Fourier Series

In the special case where the function to be expanded is purely real then we can expand the function in terms of cosines and sines using the Euler identities.

$$\begin{aligned} c_k &= \frac{1}{T_o}\int_0^{T_o} f(\tau)\exp\left(-j\frac{2\pi}{T_o}k\tau\right)d\tau \\ c_k &= \underbrace{\frac{1}{T_o}\int_0^{T_o} f(\tau)\cos\left(\frac{2\pi}{T_o}k\tau\right)d\tau}_{a_k} - j\,\underbrace{\frac{1}{T_o}\int_0^{T_o} f(\tau)\sin\left(\frac{2\pi}{T_o}k\tau\right)d\tau}_{b_k}. \end{aligned}$$

The Fourier series expansion for the function $f(t)$ can be split up as

$$f(t) = c_0 + \sum_{i=1}^{i=\infty} c_i \exp\left(j\frac{2\pi}{T_o}it\right) + \sum_{i=-\infty}^{i=-1} c_i \exp\left(j\frac{2\pi}{T_o}it\right).$$

Upon substitution of variables $m = -n$ in the sum over negative indices we can rewrite this expression as

$$f(t) = c_0 + \sum_{i=1}^{i=\infty} c_i \exp\left(j\frac{2\pi}{T_o}it\right) + \sum_{i=1}^{i=\infty} c_{-i} \exp\left(-j\frac{2\pi}{T_o}it\right).$$

Using the Euler identities and collecting terms with cosine and sines, the series expansion can be further written as

$$\begin{aligned} f(t) &= c_0 + \sum_{i=1}^{i=\infty} \underbrace{(c_i + c_{-i})}_{a_i}\cos\left(\frac{2\pi}{T_o}it\right) + \underbrace{j\,(c_i - c_{-i})}_{b_i}\sin\left(\frac{2\pi}{T_o}it\right) \\ f(t) &= c_0 + \sum_{i=1}^{i=\infty} a_i\cos\left(\frac{2\pi}{T_o}it\right) + b_i\sin\left(\frac{2\pi}{T_o}it\right) \\ f(t) &= c_0 + \sum_{i=1}^{i=\infty} \sqrt{a_i^2+b_i^2}\left[\frac{a_i}{\sqrt{a_i^2+b_i^2}}\cos\left(\frac{2\pi}{T_o}it\right) - \frac{-b_i}{\sqrt{a_i^2+b_i^2}}\sin\left(\frac{2\pi}{T_o}it\right)\right] \\ f(t) &= c_0 + \sum_{i=1}^{i=\infty} r_i\cos\left(\frac{2\pi}{T_o}it+\theta_i\right) = \sum_{i=0}^{i=\infty} r_i\cos\left(\frac{2\pi}{T_o}it+\theta_i\right). \end{aligned}$$

The coefficients in this real expansion can be related to the coefficients $\{c_k\}$ as

$$r_k = |c_k| = \sqrt{a_k^2+b_k^2}\ ,\ \theta_k = \arg(c_k) = \arctan\left(\frac{-b_k}{a_k}\right), r_0 = c_0\ ,\ \theta_0 = 0.$$

The coefficients $\{a_k\}$ and $\{b_k\}$ can also be computed directly as

$$\begin{aligned} a_k &= \frac{2}{T_o}\int_0^{T_o} f(\tau)\cos\left(\frac{2\pi}{T_o}k\tau\right)d\tau \quad , \quad k \neq 0 \\ b_k &= \frac{2}{T_o}\int_0^{T_o} f(\tau)\sin\left(\frac{2\pi}{T_o}k\tau\right)d\tau \quad , \quad k \neq 0 \\ a_0 &= \frac{1}{T_o}\int_0^{T_o} f(\tau)d\tau \end{aligned}$$