Binomial Probability Law

Consider a sequence of independent events $X_i, i = 1, 2, \ldots, n$ that are binary valued, i.e., with probabilities:

$$\Pr(X_i = 0) = 1 - p, \quad \Pr(X_i = 1) = p, \quad i = 1, \ldots, n.$$

The first possibility will be refered to as the *probability of a failure* and the second will be called the *probability of a success* on any given trial. The probability of getting k successes in these n independent trials is denoted $p_n[k]$ and is given by:

$$p_n[k] = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}.$$

Heuristically this is easy to see since, $p^k(1-p)^{n-k}$ is the probability associated with each possibility where there are k successes in n trials without ordering and the binomial coefficient is the number of such arrangements of k successes in n trials without ordering. Some of the pertinent results can be obtained directly as special cases of the classical binomial theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Specifically when $a = b = 1$, we have the familiar result:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

which corresponds to the number of possible minterms in a n-bit binary truth table. Specifically when $a = -1, b = 1$ we have the result:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

This means that sum of the even binomial coefficients is the same as the sum of the odd coefficients. Specifically when $a = p, b = 1 - p$, we obtain the result:

$$\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^{n} p_n[i] = 1.$$

This result is merely a fancy restatement of the fact that $\Pr(S) = 1$.