Discrete Random Variables

Let X be a discrete random variable that takes integer values $x_i \in \mathbf{I}$ with probabilities $p_i, i \in \mathbf{I}$. The PDF and the CDF of the discrete random variable X in this case take the general form:

$$\begin{aligned} f_X(x) &= \sum_{k=-\infty}^{\infty} p_k \delta(x-k) \\ F_X(x) &= \sum_{k=-\infty}^{\infty} p_k u(x-k). \end{aligned}$$

1. **Expectation of X**
   The mean or expected value of the random variable X is defined via:

   $$\mu_x = E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx$$

   If we substitute the special form of the PDF of the discrete random variable X into this expression we obtain the simplified relation:

   $$E(X) = \sum_{i=-\infty}^{\infty} x_i p_i$$

2. **Variance of X**
   The variance or average power of the random variable X is defined via:

   $$\sigma_X^2 = E\{(X-\mu_X)^2\} = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_X(x) dx.$$

   If we again substitute the special form of the PDF $f_X(x)$ into this expression we obtain:

   $$V(X) = \sum_{i=-\infty}^{\infty} (x_i - \mu_x)^2 p_i = \sum_{i=-\infty}^{\infty} i^2 p_i - (\mu_x)^2$$

3. **Characteristic function of X**
   The characteristic function of the random variable X is defined via:

   $$\Psi_X(j\omega) = E\{e^{j\omega X}\} = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx.$$

   Again if we substitute the specific form of the PDF into this expression we obtain:

   $$\Psi_X(e^{j\omega}) = E(e^{j\omega X}) = \sum_{i=-\infty}^{\infty} e^{j\omega i} p_i$$

   In other words the characteristic function $\Psi_x(e^{j\omega})$ for discrete random variables is just the DTFT of the probability mass sequence.

Example

Let X be Binomial random variable with parameters n, p. Calculate $E(X), V(X)$ and $\Psi_X(\omega)$. Here X can take values $1, 2, \ldots, n$ with probabilities

$$p_i = P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$\begin{aligned}
E(X) &= \sum_{i=1}^{n} i p_i = \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \qquad (1)\\
&= \sum_{i=1}^{n} i \frac{n!}{i!(n-1)!} p^i (1-p)^{n-i} \\
&= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} \\
&= np.1 = np
\end{aligned}$$

$$\begin{aligned}
V(X) &= \sum_{i=1}^{n} i^2 p_i - (np)^2 = \sum_{i=1}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i} - n^2 p^2 \qquad (2)\\
&= \sum_{i=1}^{n} (i^2 - i + i) \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} - n^2 p^2 \\
&= \sum_{i=1}^{n} \frac{(n-2)!}{(i-2)!(n-i)!} n(n-1) p^i (1-p)^{n-i} + np - n^2 p^2 \\
&= n(n-1)p^2 \sum_{i=1}^{n} \binom{n-2}{i-2} p^{i-2} (1-p)^{n-i} + np - n^2 p^2 \\
&= n(n-1)p^2 + np - n^2 p^2 = np(1-p)
\end{aligned}$$

$$\begin{aligned}
\Psi_X(\omega) &= \sum_{i=1}^{n} e^{j\omega i} p_i = \sum_{i=1}^{n} e^{j\omega i} \binom{n}{i} p^i (1-p)^{n-i} \qquad (3)\\
&= \sum_{i=1}^{n} \binom{n}{i} \left(\frac{e^{j\omega} p}{1-p} \right)^i (1-p)^n \\
&= \left(1 + \frac{e^{j\omega} p}{1-p} \right)^n (1-p)^n \\
&= (1 - p + e^{j\omega} p)^n
\end{aligned}$$

Note that in this example it would easier to evaluate the mean and variance of the random variable from the characterisitc function rather than direct evaluation via the definitions.