Appendix: Gaussian Integral

In this section we will evaluate the Gaussian Integral that will be used in various forms in the course. Let us begin by looking at the Gamma function defined by

$$\Gamma(t) = \int_0^\infty \exp(-x) x^{t-1} dx. \tag{1}$$

The Gaussian integral that we desire is a special case of the Gamma function with the substitution $x = u^2$ and $t = \frac{1}{2}$:

$$\mathbf{I} = \int_{-\infty}^{\infty} \exp\left(-x^2\right) dx = \Gamma\left(\frac{1}{2}\right), \tag{2}$$

To evaluate this integral $\mathbf{I}$, let us look at the quantity

$$\mathbf{I}^2 = \int_{-\infty}^{\infty} \exp\left(-x^2\right) dx \int_{-\infty}^{\infty} \exp\left(-y^2\right) dy \tag{3}$$

This relation can be rewritten as:

$$\mathbf{I}^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-x^2 - y^2\right) dxdy$$

$$\mathbf{I}^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-(x^2 + y^2)\right) dxdy$$

Converting the coordinate system to a polar system we have

$$\mathbf{I}^2 = \left(\frac{1}{2}\right) \int_0^{2\pi} \int_0^{\infty} \exp\left(-r^2\right) (2rdr) d\theta \tag{4}$$

Completing the evaluation we have

$$\mathbf{I}^2 = \pi \longleftrightarrow \mathbf{I} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{5}$$

This relation can then be generalized to the identity:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-x^2/2\right) dx = 1 \tag{6}$$

This identity will be used in its different forms for various other Gaussian integrals based on completing the squares. It is left as an exercise for the reader to show that

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1.$$