Independence and Uncorrelatedness

On the surface these two statistical notions look identical, however, there is a significant difference between the two. Independence of two random variables X, Y defined on the sample space $\mathbf{S}$ is characterized by the requirement that:

$$f_{X,Y}(x,y) = f_X(x)f_y(y).$$

Uncorrelatedness of the two random variables is characterized by the relation:

$$\sigma_{xy} = 0 \longleftrightarrow r_{xy} = \mu_x\mu_y.$$

Our goal in this exercise is to study the interdependence of these concepts. Suppose the two random variables are independent. Let us now compute the correlation between the random variables:

$$r_{xy} = E\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy.$$

If we now exploit the fact that the random variables are independent this integral expression can be rewritten as:

$$r_{xy} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy.$$

Since the integrands, i.e., the expression inside the integral is separable in the variables x, y this expression can be split into two separate integrals as:

$$r_{xy} = \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy = \mu_x\mu_y.$$

This in turn implies that the covariance between the random variables is :

$$\sigma_{xy} = E\{(X-\mu_x)(Y-\mu_y)\} = E\{XY\} - 2\mu_x\mu_y + \mu_x\mu_y = r_{xy} - \mu_x\mu_y = 0$$

The gist of this derivation is that *for any pair of random variables X and Y independence implies uncorrelatedness.* The converse of this statement is however not true, i.e., uncorrelatedness does not imply independence in general.