Bivariate Gaussian random variable

Let X be a bivariate gaussian random variable, i.e., $X \sim N(\mu, \Sigma)$. Denote the components of X as

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

and components of μ and Σ as

$$\mu = E(X) = E\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

and

$$\Sigma = \text{Cov}(X) = \begin{pmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

where one must understand that $\sigma_{12} = \sigma_{21}$. The PDF of X, i.e., the joint PDF of the components X_1 and X_2 is given by

$$f_X(\mathbf{x}) = \frac{1}{2\pi|\det(\Sigma)|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)\right)$$

Note that $\det(\Sigma) = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}$ and

$$\Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$$

Properties of bivariate gaussian random variable:

- It is completely parameterized by μ and Σ.
- The correlation coefficient between X_1 and X_2 is $\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$. Then the covariance matrix can be rewritten as

  $$\Sigma = \begin{pmatrix} \sigma_{11} & \rho\sqrt{\sigma_{11}\sigma_{22}} \\ \rho\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{22} \end{pmatrix}$$

  by noting that $\sigma_{12} = \sigma_{21}$. Calculating $\det(\Sigma)$ and Σ^{-1} in terms of ρ, we get $\det(\Sigma) = \sigma_{11}\sigma_{22}(1 - \rho^2)$ and

  $$\Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22}(1 - \rho^2)} \begin{pmatrix} \sigma_{22} & -\rho\sqrt{\sigma_{11}\sigma_{22}} \\ -\rho\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{11} \end{pmatrix}$$

  Another way to write the density of bivariate gaussian random variable is in terms of ρ by substituting the above formulae on $\det(\Sigma)$ and Σ^{-1}.
- If the components of X are independent then the covariance matrix Σ is a diagonal matrix and as shown in class, the joint pdf of the components X_1 and X_2 is the product of the marginals.
- In this particular case, however, uncorrelated components of X also implies independence. This is property is not true in general.