Transformation of Random Vectors:

Our goal in this section is to develop analytical results for the *probability distribution function* (PDF) of a transformed random vector $\mathbf{Y}$ *in* $\mathbf{R}^n$ given that we know the PDF, $f_{\mathbf{X}}(\mathbf{x})$, of the original random vector $\mathbf{X}$. We shall accomplish this by looking at the $n = 2$ case and then generalize the results.

Consider the sample space $\mathbf{S}_1$ defined on $\mathbf{R}^2$. $X_1(\lambda)$ and $X_2(\lambda)$ are two random variables defined on $\mathbf{S}_1$. Let $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ be two continuous and differentiable functions whose domain is the set $\mathbf{S}_1$. These functions g_1 and g_2 transform $\{X_1, X_2\}$ via:

$$\begin{aligned} Y_1 &= g_1(X_1, X_2) \\ Y_2 &= g_2(X_1, X_2). \end{aligned} \tag{1}$$

The transformed random variables $\{Y_1, Y_2\}$ belong to the transformed sample space denoted by $\mathbf{S}_2$. The infinitesimal area element in $\mathbf{S}_1$ is dx_1dx_2 and is related to the infinesimal area element in the $\mathbf{S}_2$ via the Jacobian as :

$$dx_1dx_2 = dy_1dy_2 / \left| \mathbf{J}\left(\frac{y_1, y_2}{x_1, x_2}\right) \right|, \tag{2}$$

where the Jacobian matrix is defined by:

$$\mathbf{J}\left(\frac{y_1, y_2}{x_1, x_2}\right) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} \tag{3}$$

Let us assume that the infinitesimal element in the transformed space transfers over into k infinitesimal elements in the original plane. The transformations g_1 and g_2 affect only the definition of the set $\mathbf{S_2}$. The probability measure associated with the infinitesimal element in the $\mathbf{S}_2$ must be the same as the sum of the probability measures associated with the k corresponding infinitesimal elements in $\mathbf{S}_1$. Let $\{x_1^{(i)}, x_2^{(i)}\}$ denote the i^{th} root pair of Eq. (1). Using the mapping of the probabilities over the k infinitesimal elements in the original and transformed element we can see that

$$f_{Y_1,Y_2}(y_1, y_2)\, dy_1dy_2 = \sum_{i=1}^{k} f_{X_1,X_2}\left(x_1^{(i)}, x_2^{(i)}\right) dx_1^{(i)} dx_2^{(i)}. \tag{4}$$

If we further incorporate Eq. (2) into Eq. (4) and cancel the common area term dy_1dy_2 from both sides we have the relation:

$$f_{Y_1,Y_2}(y_1, y_2) = \sum_{i=1}^{k} f_{X_1,X_2}\left(x_1^{(i)}, x_2^{(i)}\right) / \left| \mathbf{J}\left(\frac{y_1, y_2}{x_1^{(i)}, x_2^{(i)}}\right) \right|. \tag{5}$$

Generalizing this result to the random vectors $\mathbf{X}, \mathbf{Y} \in \mathbf{R}^n$ we have:

$$f_{\mathbf{Y}}(\mathbf{y}) = \sum_{i=1}^{k} f_{\mathbf{X}}(\mathbf{x}^{(i)}) / \left| \mathbf{J}\left(\frac{\mathbf{y}}{\mathbf{x}^{(i)}}\right) \right|. \tag{6}$$

The horizontal bars around the Jacobian indicate that we are taking the absolute value of the determinant of the Jacobian with an understanding that this is done so that the transformed area element is positive.

Example

Let us consider the specific case of a linear transformation of a pair of random variables defined by:

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \mathbf{b} = \begin{pmatrix} g_1(X_1, X_2) \\ g_2(X_1, X_2) \end{pmatrix} \tag{7}$$

In this problem we assume that the matrix $\mathbf{A}$ is invertible, i.e., $\mathbf{A}$ is full-rank or $\det(\mathbf{A}) \neq 0$. In this case there is only one unique root for Eq. (7) which is given by:

$$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = \mathbf{A}^{-1} \left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \mathbf{b} \right] \tag{8}$$

The Jacobian of this transformation is given by

$$\mathbf{J}\left(\frac{y_1, y_2}{x_1, x_2}\right) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \mathbf{A} \tag{9}$$

Using the results of Eq. (5) we have:

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{|\det(\mathbf{A})|} f_{X_1,X_2}\left(\mathbf{A}^{-1}[\mathbf{y} - \mathbf{b}]\right) \tag{10}$$

The marginal PDF's of the variables Y_1, Y_2 are given by :

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} \frac{1}{|\det(\mathbf{A})|} f_{X_1,X_2}\left(\mathbf{A}^{-1}[\mathbf{y} - \mathbf{b}]\right) dy_2 \\ f_{Y_2}(y_2) &= \int_{-\infty}^{\infty} \frac{1}{|\det(\mathbf{A})|} f_{X_1,X_2}\left(\mathbf{A}^{-1}[\mathbf{y} - \mathbf{b}]\right) dy_1 \end{aligned} \tag{11}$$

A special case is when $a_{11} = a_{12} = a_{21} = 1, a_{22} = 0$, $\mathbf{b} = \mathbf{0}$ and X_1, X_2 are independent RVs. In this case the marginal of Y_1 assumes the form :

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1}(y_2) f_{X_2}(y_1 - y_2) dy_2 \tag{12}$$

The above integral can be recognized as the convolution integral of the marginal PDFs of X_1 and X_2. A consequence of this is the following relation in terms of characteristic functions:

$$\Psi_{Y_1}(\omega) = \Psi_{X_1}(\omega)\Psi_{X_2}(\omega). \tag{13}$$

Transformation of Gaussian Random Vectors

Consider the case of n-variate Gaussian random vector with mean vector $\mathbf{m}_X$, covariance matrix $\mathbf{C}_X$ and PDF given by:

$$f_{\mathbf{X}}(\mathbf{x}) \;=\; \left[\frac{1}{(2\pi)^{\frac{n}{2}}|\det(\mathbf{C}_X)|^{\frac{1}{2}}}\right] \exp\left[-\frac{1}{2}\left(\mathbf{x}-\mathbf{m}_X\right)^T \mathbf{C}_X^{-1}\left(\mathbf{x}-\mathbf{m}_X\right)\right]. \tag{14}$$

This distribution is a very special one in the sense that it is completely determined by the pair: $\mathbf{m}_X, \mathbf{C}_X$. Furthermore, the distribution is only dependent on first and second-order statistics. This means that in the special case of the Gaussian random-vector uncorrelatedness of random variables corresponds to statistical independence.

Now consider a linear transformation of the random vector $\mathbf{X}$, i.e., $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$. This fits exactly into the framework of the previous example. The PDF of the transformed vector $\mathbf{Y}$ can then be evaluated using Eq. (6) as:

$$f_{\mathbf{Y}}(\mathbf{y}) \;=\; \frac{1}{|\det(\mathbf{A})|} f_{\mathbf{X}}\left(\mathbf{A}^{-1}\left[\mathbf{y}-\mathbf{b}\right]\right)$$

Substituting Eq. (14) into this result we have

$$f_{\mathbf{Y}}(\mathbf{y}) \;=\; \left[\frac{1}{(2\pi)^{\frac{n}{2}}|\det(\mathbf{A}\mathbf{C}_X\mathbf{A}^T)|^{\frac{1}{2}}}\right] \exp\left[-\frac{1}{2}\left(\mathbf{A}^{-1}(\mathbf{y}-\mathbf{b})-\mathbf{m}_X\right)^T \mathbf{C}_X^{-1}\left(\mathbf{A}^{-1}(\mathbf{y}-\mathbf{b})-\mathbf{m}_X\right)\right] \tag{15}$$

This expression can then be rewritten via linear-algebra identities as:

$$f_{\mathbf{Y}}(\mathbf{y}) = \left[\frac{1}{(2\pi)^{\frac{n}{2}}|\det(\mathbf{A}\mathbf{C}_X\mathbf{A}^T)|^{\frac{1}{2}}}\right] \exp\left[-\frac{1}{2}\left(\mathbf{y}-\mathbf{A}\mathbf{m}_X-\mathbf{b}\right)^T\left(\mathbf{A}\mathbf{C}_X\mathbf{A}^T\right)^{-1}\left(\mathbf{y}-\mathbf{A}\mathbf{m}_X-\mathbf{b}\right)\right]. \tag{16}$$

Eq. (16) shows that the random vector $\mathbf{Y}$ is also a n-variate Gaussian random vector with mean vector and covariance matrix parameters given by:

$$\begin{aligned} \mathbf{m}_Y &= \mathbf{A}\mathbf{m}_X + \mathbf{b} \\ \mathbf{C}_{\mathbf{Y}} &= \mathbf{A}\mathbf{C}_X\mathbf{A}^T. \end{aligned} \tag{17}$$

Consequently we have the important result that: "A linear transformation of a Gaussian random vector produces another Gaussian random vector". This result will be extremely useful when we consider the transmission of Gaussian random signals through linear systems. All that is needed to determine the statistics at the output of the system will be the pair in Eq. (17). Another noteworthy observation is that the factor inside the exponent of the multivariate Gaussian, i.e.,

$$D^2(\mathbf{x}, \mathbf{m}_X) \;=\; \left(\mathbf{y}-\mathbf{m}_X\right)^T \mathbf{C}_X^{-1}\left(\mathbf{y}-\mathbf{m}_X\right). \tag{18}$$

is the squared weighted Euclidean distance between the vector $\mathbf{y}$ and the mean vector $\mathbf{m}_X$.