Order Statistics

Let X_1, X_2 be independent random variables defined on a sample space $\mathbf{S} \subseteq \Re^2$. In this exercise our goal is to determine the order statistics associated with these variables. Towards this goal let us first define two random variables Y_1, Y_2 via:

$$\begin{aligned} Y_1 &= \min(X_1, X_2), \\ Y_2 &= \max(X_1, X_2). \end{aligned}$$

As a first step, we will determine the joint PDF of the variables Y_1, Y_2 via the Jacobian method as:

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1, y_2) + f_{X_1,X_2}(y_2, y_1), \quad y_1 \leq y_2.$$

Invoking the independence of the random variables X_1, X_2 the joint PDF can be written as:

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1}(y_1)f_{X_2}(y_2) + f_{X_1}(y_2)f_{X_2}(y_1), \quad y_1 \leq y_2.$$

If these random variables are further identically distributed, i.e.,

$$f_{X_1}(.) = f_{X_2}(.) = f_X(.)$$

the expression for the joint PDF reduces to:

$$f_{Y_1,Y_2}(y_1, y_2) = 2f_X(y_1)f_X(y_2), \quad y_1 \leq y_2.$$

The marginal distributions for the variables Y_1 and Y_2 are obtained via:

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{y_1}^{\infty} 2f_X(y_1)f_X(\rho)d\rho = 2f_X(y_1)\left(1 - F_X(y_1)\right) \\ f_{Y_2}(y_2) &= \int_{-\infty}^{y_2} 2f_X(y_2)f_X(\rho)d\rho = 2f_X(y_2)F_X(y_2). \end{aligned}$$

Note that the random variables Y_1, Y_2 are not independent because the joint PDF does not factor:

$$f_{Y_1,Y_2}(y_1, y_2) \neq f_{Y_1}(y_1)f_{Y_2}(y_2).$$

The marginal CDF's of the random variables Y_1, Y_2 are determined via the set-equivalence method as:

$$\begin{aligned} F_{Y_2}(y_2) &= \Pr\{\max(X_1, X_2) \leq y_2\} = [F_X(y_2)]^2, \\ F_{Y_1}(y_1) &= \Pr\{\min(X_1, X_2) \leq y_1\} = 1 - (1 - F_X(y_1))^2. \end{aligned}$$

The derivative of these expressions can be seen to be equivalent to the marginal PDF's derived before.