Examples on Transformations of Random Variables

1. Let $X \sim U([-\pi, \pi])$. Find the distribution of the random variable $Y = \cos X$.

   The density of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & \text{if } x \in [-\pi, \pi] \\ 0 & \text{otherwise} \end{cases}$$

Method 1: Note that the range of random variable Y is $[-1, 1]$. There are two solutions to the equation $y = \cos x$ for $x \in [-\pi, \pi]$, one in $[-\pi, 0]$ and the other in $[0, \pi]$. Hence, the density of $Y = \cos X$ is given by

$$\begin{aligned} f_Y(y) &= \sum_{\cos x = y} f_X(\cos^{-1} y) \left| \frac{dx}{dy} \right| \\ &= \sum_{\cos x = y} f_X(\cos^{-1} y) \left| \frac{1}{-\sin(\cos^{-1} y)} \right| \\ &= \begin{cases} \frac{2}{2\pi} \frac{1}{\sin(\cos^{-1} y)} & \text{if } \cos^{-1} y \in [0, \pi] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{\pi \sin(\cos^{-1} y)} & \text{if } y \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

If we further use the fact that $\sin(\cos^{-1}(y)) = \sqrt{1 - y^2}$ we obtain the following expression for the PDF:

$$f_Y(y) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}} & \text{if } y \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Method 2: The CDF of X is

$$F_X(x) = \frac{x - a}{b - a} = \frac{x + \pi}{2\pi} = \frac{x}{2\pi} + \frac{1}{2}$$

The sets that are equivalent to the event $\{Y \leq y\}$ are $\{X < -\cos^{-1}(y)\}$ and $\{X > \cos^{-1}(y)\}$. The CDF of Y is given by

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\cos X \leq y) \\ &= \begin{cases} 0 & y \in [-\infty, -1) \\ P(X \leq -\cos^{-1} y) + P(X \geq \cos^{-1} y) & \text{if } y \in [-1, 1] \\ 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & y \in [-\infty, -1) \\ 1 - \frac{\cos^{-1} y}{\pi} & \text{if } y \in [-1, 1] \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

assuming that $\cos^{-1} y \geq 0$. The probability density function of Y is obtained as the derivative of this CDF expression.

2. **Square law** : Let $X \sim U([-1,1])$. Find the distribution of the random variable $Y = X^2$.

   The PDF of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

   Method 1: Note that the range of random variable Y is $[0,1]$. There are two solutions to the equation $y = x^2$. Hence, the density of $Y = X^2$ is given by

$$\begin{aligned} f_Y(y) &= \sum_{x^2=y} f_X(x) \left|\frac{dx}{dy}\right| \\ &= \frac{1}{2}\left|\frac{1}{-2\sqrt{y}}\right| + \frac{1}{2}\left|\frac{1}{2\sqrt{y}}\right| \\ &= \begin{cases} \frac{1}{2\sqrt{y}} & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3}$$

   Method 2: The CDF of X is

$$F_X(x) = \frac{x}{2}$$

   The CDF of Y is given by

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) \\ &= \begin{cases} 0 & y \in [-\infty, 0) \\ P(-\sqrt{y} \leq X \leq \sqrt{y}) & \text{if } y \in [0,1] \\ 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & y \in [-\infty, -1) \\ \frac{\sqrt{y}}{2} - \frac{-\sqrt{y}}{2} = \sqrt{y} & \text{if } y \in [-1,1] \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

   Hence, the density of Y is given by (2).

3. **Full-wave rectifier**: Let $X \sim U([-1,1])$. Find the distribution of the $Y = Xu(X)$.

   The density of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

   Note that this transformation is not differentiable so the gradient method is not applicable in this case. Instead let us look at the set equivalence method.

- For $y < 0$, the event $\{Y \leq y\}$ does not have a solution on the real line and hence reduces to a null event. Consequently the probability of this event is 0.
- For $y = 0$, the event $\{Xu(X) \leq 0\}$ has only one solution $X = 0$ and the probability of this event is $F_X(0_+) - F_X(0_-)$.
- For $y > 0$, the event that $\{Xu(X) \leq y\}$ reduces to the event $\{X \leq y\}$ and the probability of this event is just $F_X(y)$.

The CDF of the transformed random variable can then be summarized as:

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(0_+) - F_X(0_-) & y = 0 \\ F_X(y) & y > 0 \end{cases}$$

Since the input distribution is uniform this reduces to:

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ F_X(y) & y > 0 \end{cases}$$