ECE - 439, Fall 2011
Intro to DSP

Example: DTFT of Periodic Signals

Consider the discrete sequence

$$x[n] = \sum_{k=0}^{N-1} c[k] \exp\left(j \frac{2\pi}{N} nk\right) \quad (1)$$

We will consider DTFT's of the form

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} 2\pi\, c[k]\, \delta\left(\omega - k\frac{2\pi}{N} + 2r\pi\right)$$

The inverse DTFT of this expression is given by:

$$\begin{aligned}
x[n] &= \mathrm{IDTFT}\{X(e^{j\omega})\} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} 2\pi c[k]\, \delta\left(\omega - k\frac{2\pi}{N} + 2r\pi\right) e^{j\omega n}\, d\omega \\
&= \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi c[k]}{2\pi} \int_{-\pi}^{\pi} e^{j\omega n}\, \delta\left(\omega - \frac{2\pi k}{N} + 2r\pi\right) d\omega
\end{aligned}$$

Using the sifting theorem:

$$e^{j\omega n} \delta(\omega - \tfrac{2\pi}{N} k + 2r\pi)$$

$$= \exp(j(\tfrac{2\pi}{N} k - 2r\pi) n)$$
$$\delta(\omega - \tfrac{2\pi}{N} k + 2r\pi)$$

$$x[n] = \sum_{k=0}^{N-1} c[k] \sum_{r=-\infty}^{\infty} \int_{-\pi}^{\pi} \delta(\omega - k\tfrac{2\pi}{N} + 2r\pi)\, d\omega$$
$$\exp(j \tfrac{2\pi}{N} nk)$$

Since the integral is over $[-\pi, \pi]$ only the $r = 0$ term remains

$$x[n] = \sum_{k=0}^{N-1} c[k] \exp(j \tfrac{2\pi}{N} nk)$$

Of course this is the discrete sequence in (1) that we started of with.

$$x[n] = \sum_{k=0}^{N-1} c[k] \exp(j \tfrac{2\pi}{N} nk)$$

$$\mathcal{F}^{-1} \Big\Updownarrow \mathcal{F}$$

$$X(e^{j\omega}) = 2\pi \sum_{k=0}^{N-1} c[k]\, \delta(\omega - k\tfrac{2\pi}{N}),$$
$$|\omega| < \pi$$