ECE-439, Fall 2011
INTRO TO DSP

Example : Differentiation Property of Z-transforms

Consider a z-domain expression of the form:

$$X(z) = \frac{1}{(1-az^{-1})^2}, \quad \begin{matrix} |z| > |a| \\ |a| < 1 \end{matrix}$$

Let us focus on the z-transform pair:

$$x[n] = a^n u[n] \underset{z^{-1}}{\overset{\mathcal{Z}}{\rightleftharpoons}} X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a| \qquad (1)$$

The differentiation property :

$$-z \frac{dX(z)}{dz} \underset{\mathcal{Z}}{\overset{\mathcal{Z}^{-1}}{\rightleftharpoons}} n x[n]$$

$\frac{dX}{dz}$ for the pair in (1) is:

$$\frac{dX(z)}{dz} = \frac{(z-a)1 - z}{(z-a)^2}$$

$$= \frac{-a}{(z-a)^2} = \frac{-az^{-2}}{(1-az^{-1})^2}$$

$$\Rightarrow z\frac{dX(z)}{dz} = \frac{-az^{-1}}{(1-az^{-1})^2}$$

$$\Rightarrow -\frac{1}{a}z\frac{dx(z)}{dz} = \frac{+z^{-1}}{(1-az^{-1})^2}$$

$$\Rightarrow -\frac{1}{a}z^2\frac{dx}{dz} = \frac{1}{(1-az^{-1})^2}$$

We can now write down the pair :

$$\frac{1}{(1-az^{-1})^2}, |z|>|a| = -\frac{1}{a}z^2\frac{dx}{dz}$$

$$= +\frac{1}{a}z\cdot-\left(z\frac{dx}{dz}\right)$$

$$z \;\|\!\|\; z^{-1}$$

$$\frac{1}{a}(n+1)x[n+1]$$

Incorporating the expression for $x[n]$:

$$\frac{1}{(1-az^{-1})^2}, |z|>|a| \underset{\mathcal{Z}}{\overset{\mathcal{Z}^{-1}}{\rightleftharpoons}} \frac{1}{a}(n+1)a^{n+1}u[n+1]$$

$$= a^n(n+1)u[n+1]$$