Notes on Filterbanks

In class, we have seen that the Nyquist sampling theorem specifies a minimum sampling rate for a band-limited analog signal so that information is not lost during the sampling process. If we sample at a rate lower that this rate we introduce aliasing. A filterbank is a signal processing system, comprised of upsamplers, downsamplers and filters, that allows one to process the signal at a fraction of the Nyquist rate and forms the basis for subband signal compression schemes. In this note we will focus specifically on the two-channel case.

Analysis Section

Recognize that both the branches of the analysis section are zoom operations. The signal $x_0[n]$ corresponds to the lowpass subband signal and the signal $x_1[n]$ corresponds to the highpass subband signal. In the time-domain the expressions that relate the subband signals to the input signal $x[n]$ are given by:

$$\begin{aligned} x_0[n] &= \sum_{k=-\infty}^{\infty} x[k]h_0[2n-k] \\ x_1[n] &= \sum_{k=-\infty}^{\infty} x[k]h_1[2n-k] \end{aligned}$$

The corresponding relations in the Z-domain are:

$$\begin{aligned} X_0(z) &= \frac{1}{2}\left\{X(z^{0.5})H_0(z^{0.5}) + X(-z^{0.5})H_0(-z^{0.5})\right\} \\ X_1(z) &= \frac{1}{2}\left\{X(z^{0.5})H_1(z^{0.5}) + X(-z^{0.5})H_1(-z^{0.5})\right\} \end{aligned}$$

Note that both of these subband expressions have both a desired term and an aliasing term that is a direct result of the downsampling operation in each branch.

Synthesis Section

Recognize in this case that each of the branches is an interpolation operation by a factor of 2. The output of the filterbank $\hat{x}[n]$ can be related to the subband signals in the time-domain via:

$$\hat{x}[n] = \sum_{k=-\infty}^{\infty} x_0[k]f_0[n-2k] + \sum_{k=-\infty}^{\infty} x_1[k]f_1[n-2k].$$

In the Z-domain the correspoding relation is:

$$\hat{X}(z) = F_0(z)X_0(z^2) + F_1(z)X_1(z^2).$$

Substituting the expressions obtained for the subband signals in the analysis section and grouping desired terms and alias terms we attain:

$$\begin{aligned} \hat{X}(z) &= \frac{1}{2}\left\{F_0(z)H_0(z) + F_1(z)H_1(z)\right\}X(z) \\ &+ \frac{1}{2}\left\{F_0(z)H_0(-z) + F_1(z)H_1(-z)\right\}X(-z). \end{aligned}$$

The second term in this expression corresponds to the *alias component* system function:

$$A(z) = \frac{1}{2}\left\{F_0(z)H_0(-z) + F_1(z)H_1(-z)\right\} \qquad (1)$$

and the first term correspond to the *transmit component* system function:

$$T(z) = \frac{1}{2}\left\{F_0(z)H_0(z) + F_1(z)H_1(z)\right\} \qquad (2)$$

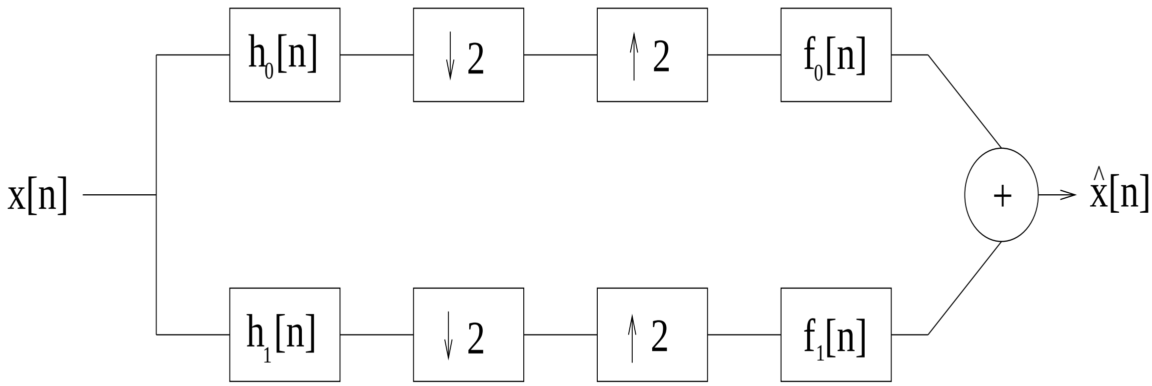

Figure 1: Two channel, maximally decimated filterbank.

The motivation behind the use of the filterbank is to design the analysis and synthesis filters so that the alias term is made zero and the desired term corresponds to a simple gain-delay system. For the cancelation of the alias component, we require that:

$$F_0(z)H_0(-z) + F_1(z)H_1(-z) = 0 \tag{3}$$

If we further require that the filterbank system correspond to a *perfect reconstruction* (PR) filterbank we require:

$$F_0(z)H_0(z) + F_1(z)H_1(z) = 2cz^{-n_o}, n_o \in \mathbf{I}. \tag{4}$$

Formulating the design equations in a matrix form we have:

$$\underbrace{\begin{pmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{pmatrix}}_{\mathbf{A(z)}} \begin{pmatrix} F_0(z) \\ F_1(z) \end{pmatrix} = \begin{pmatrix} 2cz^{-n_o} \\ 0 \end{pmatrix} \tag{5}$$

The rational matrix $\mathbf{A}$(z) is referred to as the *alias component* (AC) matrix. Note that at the present the design is incomplete because we have 4 unknown filters $H_0(z), H_1(z), F_0(z), F_1(z)$ and only 2 equations. So at present there is no unique solution to the PR filterbank design problem. However, if we focus our attention on the alias cancelation equation then one such solution for alias cancelation is:

$$H_1(z) = H_0(-z),\ F_0(z) = H_0(z),\ F_1(z) = -H_1(z). \tag{6}$$

It can be easily verified that this solution actually satisfies the alias-cancelation equation. This solution will hereafter be refered to as the *quadrature mirror filter* (QMF) solution to the alias cancelation problem. The QMF terminology follows from the observation:

$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)}). \tag{7}$$

In otherwords the highpass filter frequency response is a mirror image of the lowpass filter frequency response.