Finite Dimensional Vector Spaces

1. The inner product of two N dimensional vectors $\vec{f}$ and $\vec{g}$ in the Euclidean space $\mathbf{R}^N$ is given by

$$< \vec{f}, \vec{g} > = \sum_{i=1}^{i=N} f_i g_i,$$

where $\vec{f} = \{f_1, f_2, \ldots, f_N\}$ are the components of the vector $\vec{f}$.

2. Two vectors $\vec{f}$ and $\vec{g}$ in $\mathbf{R}^N$ are orthogonal or perpendicular if and only if

$$< \vec{f}, \vec{g} > \quad = \quad 0$$

3. The norm or length of a vector $\vec{f}$ in $\mathbf{R}^N$ denoted $||\vec{f}||_2$ is given by:

$$||f||_2 \quad = \quad \sqrt{< \vec{f}, \vec{f} >} \quad = \quad \sqrt{\sum_{i=1}^{i=N} f_i^2}$$

4. The vector $\vec{f}$ is said to be unit norm if $||f||_2 \ = \ 1$.

5. The finite collection of vectors $\{\vec{v}_i, i \ = \ 1, 2, \ldots, N\}$ in $\mathbf{R}^N$ is said to be a orthonormal set of vectors if and only if

$$< \vec{v}_i, \vec{v}_j > \quad = \quad \delta_{m,n} \quad \equiv \quad \begin{cases} 1 & m = n \\ 0 & m \neq n, \end{cases}$$

for every m and n in the collection and $\delta_{m,n}$ denotes the Kroneker delta symbol.

6. The finite collection of vectors $\{\vec{v}_i, i \ = \ 1, 2, \ldots, N\}$ forms a basis for the finite dimensional vector space $\mathbf{R}^N$ if and only if any vector $\vec{v} \ \in \ \mathbf{R}^N$ can be expressed uniquely as a linear combination of these vectors:

$$\vec{v} \quad = \quad \sum_{i=1}^{i=N} c_i v_i \quad = \quad \sum_{i=1}^{i=N} < \vec{v}, \vec{v}_i > \vec{v}_i,$$

where c_i are the expansion coefficients.

Function Spaces

1. The function $f(t)$, $t \in [a, b]$ is said to be an element of the class or family of functions $\mathbf{H}[a, b]$ that are square-integrable if and only if it satisfies:

$$E_f \quad = \quad \int_a^b |f(t)|^2 dt \quad < \quad \infty$$

   This is called the membership criteria.

2. The inner product of two functions $f(t)$ and $g(t)$ in $\mathbf{H}[a, b]$ is defined as:

$$< f(t), g(t) > \quad \equiv \quad \int_a^b f(\tau) g^*(\tau) d\tau.$$

3. The norm of a function $f(t)$ is defined by

$$\begin{aligned} ||f||^2 \quad &= \quad < f(t), f(t) > \; = \; E_f \; = \; \int_a^b |f(t)|^2 dt \\ ||f|| \quad &= \quad \sqrt{E_f} \end{aligned}$$

   The space $\mathbf{H}[a, b]$ therefore contains functions $f(t)$ that have finite norm.

4. Two functions $f(t)$ and $g(t)$ in $\mathbf{H}[a, b]$ are said to be orthogonal if:

$$< f(t), g(t) > \; = \; \int_a^b f(\tau) g^*(\tau) d\tau \; = \; 0.$$

5. The countably infinite collection of functions $\{\phi_i(t), i = 1, 2, \ldots, \infty\}$ is said to be an orthonormal collection of functions if and only if

$$< \phi_m(t), \phi_n(t) > = \int_a^b \phi_i(\tau) \phi_j^*(\tau) d\tau \; = \; \delta_{m,n} \quad \equiv \quad \begin{cases} 1 & m = n \\ 0 & m \neq n, \end{cases}$$

   for every m and n of the collection and $\delta_{m,n}$ denotes the Kroneker delta symbol.

6. The countably infinite collection of functions $\{\phi_i(t), i = 1, 2, \ldots, \infty\}$ is said to be a basis for $\mathbf{H}[a, b]$ if and only if every $f(t) \in \mathbf{H}[a, b]$ on the interval $t \in [a, b]$ can be expanded uniquely as a linear combination of these functions as

$$f(t) \quad = \quad \sum_{i=1}^{i=\infty} c_i \phi_i(t),$$

   where c_i are the expansion coefficients.