ECE-495, FALL 2011

INTRO TO DSP

Example: Nyquist - Shannon Wavelets.

Consider the ideal LPF with Gain T_s and cut-off frequency $\Omega_c = \frac{\pi}{T_s}$

$$H_{LP}(j\Omega) = \begin{cases} T_s, & |\Omega| < \frac{\pi}{T_s} \\ 0, & \text{otherwise} \end{cases}$$

The impulse response correspondingly is

$$h_{LP}(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right)$$

Now consider a sequence of functions generated from shifts of the lowpass filter response:

$$\phi_k(t) = h_{LP}(t - kT_s), \quad -\infty \leq k \leq \infty$$

Sampling theorem states that if the prescriptions of the theorem are satisfied:

$$\hat{x}_c(t) = \sum_{k=-\infty}^{\infty} x_c(kT_s) \operatorname{sinc}\left(\frac{t}{T_s} - k\right)$$

Further the pair-wise inner product between elements of the sequence is:

$$\langle \phi_p(t), \phi_q(t) \rangle = \int_{-\infty}^{\infty} \phi_p(t)\, \phi_q^*(t)\, dt$$

Using Parsevol's theorem

$$\langle \phi_p(t), \phi_q(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{LP}(j\Omega)\, e^{-j\Omega p T_s}\, H_{LP}^*(j\Omega)\, e^{j\Omega q T_s}\, d\Omega$$

$$\langle \phi_p(t), \phi_q(t) \rangle = \frac{T_s^2}{2\pi} \int_{-\pi/T_s}^{\pi/T_s} e^{-j\Omega(p-q)T_s}\, d\Omega$$

$$\langle \phi_p(t), \phi_q(t) \rangle = \begin{cases} T_s & ; \; p = q \\ 0 & ; \; p \neq q \end{cases}$$

$$= T_s\, \delta_{pq}$$

This implies that $\{\phi_k(t)\}_{k=-\infty}^{\infty}$ is a orthogonal sequence

In light of the sinc reconstruction formula $\{\phi_k(t)\}_{k=-\infty}^{\infty}$ constitutes a orthogonal basis for $x_c(t) \ni X_c(j\Omega) \in L^2([-\frac{\pi}{T_s}, \frac{\pi}{T_s}])$

The prototype $h_{LP}(t)$ is the scaling function and the elements of $\{\phi_k(t)\}$ are called the Nyquist-shannon wavelets