PS #3 , Spring 2001
Signal Processing Using MATLAB, EECE-495
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MATLAB Assignment
Date Assigned: 02/07/2001
Date Due: 02/13/2001

Background

The goal of this exercise is to design a discrete–time FIR filter using the least–squares technique. For a discrete–time, LTI system with impulse response $h[n]$, we define the frequency response of the system via:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \exp(-j\omega n).$$

For the specific case where the filter is a *finite impulse response* (FIR) filter, we have simplify the above as:

$$H(e^{j\omega}) = \sum_{n=0}^{L-1} h[n] \exp(-j\omega n).$$

This quantity however, is still an non computable quantity because the frequency variable ω is still a continuous variable defined on $[-\pi, \pi]$. Instead if we sampled the frequency grid at $\omega_k,\ 0 \leq k \leq N-1$ we have :

$$H(e^{j\omega_k}) = \sum_{n=0}^{L-1} h[n] \exp(-j\omega_k n),\ 0 \leq k \leq N-1.$$

This quantity is a computable quantity because it can be written as the inner product of two vector via:

$$H(e^{j\omega_k}) = [1\ e^{-j\omega_k}\ e^{-j2\omega_k} \ldots e^{-j(L-1)\omega_k}]\mathbf{h},$$

where $\mathbf{h} = [h[0]\ h[1]\ \ldots h[L-1]]^T$ is a vector containing the impulse response coefficients. Rearranging these constraints in the form of a linear system of equations we have:

$$\begin{pmatrix} 1 & e^{-j\omega_1} & e^{-j2\omega_1} & \dots e^{-j(L-1)\omega_1} \\ 1 & e^{-j\omega_2} & e^{-j2\omega_2} & \dots e^{-j(L-1)\omega_2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-j\omega_N} & e^{-j2\omega_N} & \dots e^{-j(L-1)\omega_N} \end{pmatrix} \begin{pmatrix} h[0] \\ h[1] \\ \vdots \\ h[L-1] \end{pmatrix} = \begin{pmatrix} H(e^{j\omega_1}) \\ H(e^{j\omega_2}) \\ \vdots \\ H(e^{j\omega_N}) \end{pmatrix},$$

where we have assumed that $N < \frac{L+1}{2}$. Note that these coefficients are in general complex. For implementation purposes if we assume that the LTI system is a *type I*, FIR system, i.e., L is odd and further constrain the filter $h[n]$ coefficients to be symmetric, i.e.,

$$h[n] = h\left[L-1-n\right], \quad n = 0, 1, \dots, \frac{L-1}{2}.$$

The frequency response relation can then be rewritten in the form of:

$$\boxed{H(e^{j\omega_k}) = \sum_{n=0}^{L-1} h[n] \exp(-j\omega_k n) = e^{-j\omega_k\left(\frac{L-1}{2}\right)} \sum_{n=0}^{\frac{L-1}{2}} a[n] \cos(\omega_k n),} \tag{1}$$

where

$$\boxed{a[0] = h\left[\frac{L-1}{2}\right], \; a[n] = 2h\left[\frac{L-1}{2} - n\right], \; n = 1, 2, \dots, \frac{L-1}{2}.} \tag{2}$$

These $\frac{L+1}{2}$ equations in the seuqnce $a[n]$ can then be rearranged in the following matrix form:

$$\underbrace{\begin{pmatrix} 1 & \cos\omega_1 & \cos 2\omega_1 & \dots \cos\frac{(L-1)\omega_1}{2} \\ 1 & \cos\omega_2 & \cos 2\omega_2 & \dots \cos\frac{(L-1)\omega_2}{2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cos\omega_N & \cos 2\omega_N & \dots \cos\frac{(L-1)\omega_N}{2} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} a[0] \\ a[1] \\ \vdots \\ a\left[\frac{L-1}{2}\right] \end{pmatrix}}_{\mathbf{a}} = \underbrace{\begin{pmatrix} |H(e^{j\omega_1})| \\ |H(e^{j\omega_2})| \\ \vdots \\ |H(e^{j\omega_N})| \end{pmatrix}}_{\mathbf{b}},$$

The matrix $\mathbf{A}$ in the above system has full row–rank, i.e., $\text{rank}(\mathbf{A}) = \min(N, \frac{L-1}{2}) = N$. The solution to this system is obtained via the least-squares right inverse:

$$\boxed{\mathbf{a} = \mathbf{A}_R^{\dagger}\mathbf{b} = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{b}.} \tag{3}$$

Unlike the earlier system this equation system is a real and the solution for the filter coefficients will be real.

Design Algorithm

- For a given set of frequencies ω_k and corresponding frequency response magnitudes $|H(e^{j\omega_k})|$, generate the matrix $\mathbf{A}$ and the vector $\mathbf{b}$.
- Use Eq. (3) to obtain the coefficients $a[n]$ via least-squares inversion.
- Use Eq. (2) to obtain the impulse response coefficients $h[n]$ from the coefficients $a[n]$ obtained in the previous step.
- Compute the design error measure using:

$$e(dB) = 20log_{10}(||\mathbf{Aa} - \mathbf{b}||_2).$$

Problem Outline

1. Write a matlab function `lsfilt.m` with the following synopsis:

```
[hopt,edB] = lsfilt(freq,magn,ord)
hopt: impulse response vector
edB : design error in dB
freq: vector of frequency points
magn: vector of magnitude responses
ord : order of desired FIR filter
```

2. Specifically implement the least squares system using least–squares inversion or matrix right division in MATLAB, i.e,

```
>> a_vec = A/b;
```

3. Include appropriate error checking for the function. Specifically check to see if frequencies in the vector `freq` have to lie in $[0, \pi]$ and the magnitudes in the vector `magn` have to be positive.

4. Test the function using the input:

```
>> freq = [0 pi/2 pi]; magn = [0 1 0];
>> freq = [0 pi/4 pi]; magn = [1 0.707 0];
```

5. For each case plot the freuency response of the designed filter using the function `freqz.m`.