Problem Set # 6.0
Signal Processing Exercises in MATLAB
Instructor: Balu Santhanam
EECE–495, Spring 2001
Assigned: 04/09/01, Due: 04/16/01

Background:

As discussed in class, design of a filterbank involves the solution to the 4 parameters $H_o(z)$, $H_1(z)$, $F_o(z)$ and $F_1(z)$. In this section we will look at the design of specific filterbanks that will allow us to cancel the alias introduced by the downsampling operations or filters that will allow us to achieve perfect reconstruction.

Quadrature mirror filterbanks (QMF) are filterbanks designed with the constraint

$$H_1(z) = H_o(-z) \Longleftrightarrow H_1(e^{j\omega}) = H_o(e^{j(\omega-\pi)}).$$

In other words the lower analysis filter is a mirror image of the upper analysis filter. Perfect reconstruction filters on the otherhand are filters that satisfy:

$$\hat{X}_1(z) = cz^{-n_o}X(z), \quad n_o \in I$$

the output of the filterbank is a scaled shifted version of the input.

If our design objective is to cancel the alias terms introduced by the downsampling operations in the analysis section then the design goal that we have to satisfy is:

$$H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0.$$

Combining the quadrature mirror constraint along with the specification of $H_o(z)$ there is still one free parameter and hence an entire line of solutions for alias cancelation. One specific solution for the alias cancelation is:

$$F_o(z) = H_o(z) \;\;\&\;\; F_1(z) = -H_o(-z).$$

If on the otherhand, the design objective is perfect reconstruction then the design equations can be reformulated as:

$$\begin{aligned} H_0(z)F_0(z) + H_1(z)F_1(z) &= cz^{-n_0} \\ H_0(-z)F_0(z) + H_1(-z)F_1(z) &= 0. \end{aligned}$$

These are two equations in three parameters $H_1(z)$, $F_0(z)$ and $F_1(z)$. Reformulating these reconstruction equations into a matrix framework :

$$\begin{pmatrix} F_0(z) & F_1(z) \end{pmatrix} \begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix} = \begin{pmatrix} cz^{-n_0} & 0 \end{pmatrix}.$$

Incorporating the quadrature mirror constraint into this framework we can then invert this equation system to obtain the analysis filters. The solution to the QMF–PR system can then be formulated as:

$$H_o(z) = E_o(z^2) + z^{-1}E_1(z^2), \qquad H_1(z) = E_o(z^2) - z^{-1}E_1(z^2)$$
$$F_o(z) = \frac{1}{E_1(z^2)} + \frac{z^{-1}}{E_0(z^2)}, \qquad F_1(z) = \frac{-1}{E_1(z^2)} + \frac{z^{-1}}{E_0(z^2)}.$$

Note that in general the synthesis filters could be IIR filters.

Program Outline

1. Write a matlab function `analy.m` that given the analysis filters $H_o(z)$ and $H_1(z)$ will implement the filter decimate sections and produces the output signals $u_o[n]$ and $u_1[n]$. The function synopsis is

```
>> [u_o,u_1] = analy(x,b_o,a_o,b_1,a_1)
% u_o, u_1 : subband outputs
% b_o      : numer coeff of H_o
% a_o      : denom coeff of H_o
% b_1      : numer coeff of H_1
% a_1      : denom coeff of H_1
```

2. Write a matlab function `synth.m` that given the synthesis filters $F_o(z)$ and $F_1(z)$ will implement the interpolation section to produce the reconstructed output $\hat{x}[n]$. The function synopsis is

```
>> x_hat = synth(u_o,u_1,b_2,a_2,b_3,a_3)
% x_hat    : reconstructed output
% u_o, u_1 : subband inputs
% b_2      : numer coeff of F_o
% a_2      : denom coeff of F_o
% b_3      : numer coeff of F_1
% a_3      : denom coeff of F_1
```

3. Implement the QMF alias cancelation solution using a FIR lowpass filter with system function: $H_0(z) = 2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}$. Plot the frequency responses of the designed QMF systems.

4. Implement the PR solution using the same filter and quadrature mirror constraints. Note that the solution for the synthesis filters could yield IIR systems. Plot the frequency responses of the filters in the designed PR system.

5. For implementing the filtering operations, you can use the function `filter.m`. For the downsample and upsample operations, you should write appropriate functions of your own.