Non Uniform Quantization

In class we looked at the additive noise model for uniform quantization, where under certain assumptions the quantized signal can be represented by the model:

$$Q(x[n]) = x_q[n] = x[n] + e_q[n],$$

where $e_q[n]$ is the quantization noise that is assumed to be statistically uncorrelated with the source signal $x[n]$. This model however, is not appropriate for the non uniform quantization process, where the quantization error is correlated with the input. In fact recall that the Lloyd-Max approach exploits the likelihood of a source value falling in the interval $[x_i, x_{i+1}]$ to determine the output alphabets m_i. Here we will look at two different models for non uniform quantization and the equivalence between them. The first model looks at a correlated noise structure of the form:

$$Q(x[n]) = x_q[n] = x[n] + e_q[n], \tag{1}$$

where the quantization error $e_q[n]$ is correlated with the source signal $x[n]$. However, the quantized output signal $x_q[n]$ is assumed to be uncorrelated with the quantization error, i.e.,

$$E\{x_q[n]e_q[n]\} = E\{e_q[n]\}E\{x_q[n]\} = 0.$$

Using this model, we can compute the correlation between the source signal $x[n]$ and the quantization error $e_q[n]$ as:

$$E\{x[n]e_q[n]\} = E\{(x_q[n] - e_q[n])e_q[n]\} = -\sigma_e^2. \tag{2}$$

The second representation that we will look at is the gain noise model of the form:

$$x_q[n] = \alpha x[n] + r[n], \tag{3}$$

where α is the model gain and $r[n]$ is the additive noise term. For equivalence in the mean in the output between these two models we have:

$$E\{x[n]\} = \alpha E\{x[n]\} + E\{r[n]\} \Longleftrightarrow (1-\alpha)E\{x[n]\} = E\{r[n]\} \tag{4}$$

The correlation between the source signal and the quantization error signal $e_q[n]$ after incorporating the gain model into it is:

$$E\{x[n]e_q[n]\} = E\{x[n](x_q[n] - x[n])\} = E\{x[n](\alpha x[n] + r[n] - x[n])\}$$

This relation can be rewritten as

$$E\{x[n]e_q[n]\} = (\alpha - 1)E\{x^2[n]\} - E\{r[n]x[n]\} = (\alpha - 1)(\sigma_x^2 + \mu_x^2) + E\{r[n]x[n]\}$$

Upon incorporating the equivalence in the mean into this relation we obtain:

$$E\{x[n]e_q[n]\} = (\alpha - 1)\sigma_x^2 - E\{r[n]\}E\{x[n]\} + E\{x[n]r[n]\} = -\sigma_e^2$$

The specific gain α that would result in no correlation between the input signal $x[n]$ and the noise term $r[n]$, just as we had in the case of uniform quantization model is given by :

$$\sigma_{rx} = E\{r[n]x[n]\} - E\{r[n]x[n]\} = 0 \Longleftrightarrow (\alpha - 1)\sigma_x^2 - \sigma_e^2 = 0.$$

Or equivalently the particular value of the gain parameter that decorrelates $x[n]$ and $r[n]$ in the gain and noise model is:

$$\alpha_g = 1 - \frac{\sigma_e^2}{\sigma_x^2}. \tag{5}$$

The other consequence of no correlation between $r[n]$ and $x[n]$ is the relation:

$$E\{x[n]e_q[n]\} = (\alpha - 1)\sigma_x^2. \tag{6}$$

For equivalence in the variance of the output between the two models when the gain is chosen as above, we look at the mean-square value of the variance at the output of the quantizer in the first model :

$$E\{x_q^2[n]\} = \alpha^2 E\{x^2[n]\} + E\{r^2[n]\} + 2\alpha E\{r[n]x[n]\} \tag{7}$$

Evaluating the same expression from the first model we have:

$$E\{x_q^2[n]\} = E\{x^2[n]\} + E\{e_q^2[n]\} + 2E\{e_q[n]x[n]\} \tag{8}$$

Substituting the expression in Eq. (6) into the expression in Eq. (8) we obtain:

$$E\{x_q^2[n]\} = E\{x^2[n]\} + E\{e_q^2[n]\} + 2(\alpha - 1)\sigma_x^2. \tag{9}$$

Substituting Eq. (5) into the expression in Eq. (9) we obtain:

$$E\{x_q^2[n]\} = E\{x^2[n]\} + (\alpha - 1)\sigma_x^2 \tag{10}$$

Comparing the expression in Eq. (10) and Eq. (7) for the same expression from both models we obtain:

$$\sigma_r^2 = \alpha(1 - \alpha)\sigma_x^2. \tag{11}$$

The significance of these relations is that the signal model for the non-uniform quantization process can be put into a gain-noise model, where the source signal and the noise term are uncorrelated in a manner similar to that of the uniform quantizer model.