ECE-539, SPRING 2009
Digital Signal Processing

EXAMPLE: Comparison of Approaches

Consider the system:

$$H(z) = \frac{1}{(1-p_1 z^{-1})(1-p_2 z^{-1})}, \quad |z| > \max[|p_1|, |p_2|]$$

Note that some forms require the computation of:

$$\sum_{n=0}^{\infty} h^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega$$

$$|H(e^{j\omega})|^2 = \frac{k_1}{1+p_1^2-2p_1\cos\omega} + \frac{k_2}{1+p_2^2-2p_2\cos\omega}$$

Comparing Coefficients:

$$k_1(1+p_2^2) + k_2(1+p_1^2) = 1$$

$$-2p_2 k_1 - 2p_1 k_2 = 0$$

Solving them simultaneously:

$$k_1 = \frac{-p_1}{(p_2-p_1)(1-p_1p_2)}, \quad k_2 = \frac{p_2}{(p_2-p_1)(1-p_1p_2)}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega = \frac{k_1}{2\pi}\int_{-\pi}^{\pi} \frac{d\omega}{1+p_1^2-2p_1\cos\omega}$$

$$+ \frac{k_2}{2\pi}\int_{-\pi}^{\pi} \frac{d\omega}{(1+p_2^2-2p_2\cos\omega)}$$

Substituting $\tan\frac{\omega}{2} = t \quad \Big| \ t : -\infty \to \infty$

$$\frac{2\,dt}{1+t^2} = d\omega$$

$$I_1 = \int_{-\infty}^{\infty} \frac{2\,dt/_{1+t^2}}{1+p_1^2 - 2p_1\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_{-\infty}^{\infty} \frac{2\,dt}{(1+p_1^2)(1+t^2) - 2p_1(1-t^2)}$$

$$= \int_{-\infty}^{\infty} \frac{2\,dt}{1+t^2+p_1^2+p_1^2t^2-2p_1+2p_1t^2}$$

$$= \int_{-\infty}^{\infty} \frac{2\,dt}{(1-p_1)^2+t^2(1+p_1)^2}$$

$$= \int_{-\infty}^{\infty} \frac{2}{(1+p_1)^2} \frac{dt}{t^2+\frac{(1-p_1)^2}{(1+p_1)^2}}$$

$$= \frac{2}{(1+p_1)^2} \frac{1}{\frac{1-p_1}{1+p_1}} \tan^{-1}\left(\frac{t}{\frac{1-p_1}{1+p_1}}\right)\Bigg|_{-\infty}^{\infty}$$

$$= 2\left(\frac{1}{1-p_1^2}\right)\pi$$

Similarly

$$I_2 \triangleq \int_{-\infty}^{\infty} \frac{2dt/_{1+t^2}}{1+p_2^2 - 2p_2\left(\frac{1-t^2}{1+t^2}\right)}$$

$$I_2 \triangleq \frac{2\pi}{1-p_2^2}$$

$$\sum_{n=0}^{\infty} h^2[n] = \frac{K_1}{1-p_1^2} + \frac{K_2}{1-p_2^2}$$

$$= \frac{p_1}{(p_1-p_2)(1-p_1p_2)(1-p_1^2)} + \frac{p_2}{(p_2-p_1)(1-p_1p_2)(1-p_2^2)}$$

$$= \frac{1}{(p_1-p_2)(1-p_1p_2)}\left[\frac{p_1}{1-p_1^2} - \frac{p_2}{1-p_2^2}\right]$$

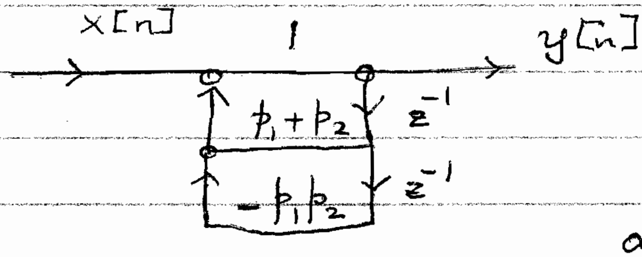
$$= \frac{1}{(p_1-p_2)(1-p_1p_2)}\left\{\frac{p_1 - p_1p_2^2 - p_2 + p_2p_1^2}{(1-p_1^2)(1-p_2^2)}\right\}$$

$$= \frac{1}{(p_1-p_2)(1-p_1p_2)}\left\{\frac{(p_1-p_2) + p_2p_1(p_1-p_2)}{(1-p_1^2)(1-p_2^2)}\right\}$$

$$= \frac{\cancel{(p_1-p_2)}(1+p_1p_2)}{\cancel{(p_1-p_2)}(1-p_1p_2)} \quad \frac{1}{(1-p_1^2)(1-p_2^2)}$$

$$= \frac{\frac{1+p_1p_2}{1-p_1p_2}}{(1-p_1^2)(1-p_2^2)}$$

Direct Form:



$$\sigma_f^2 = 2\sigma_e^2 \sum_{n=0}^{\infty} h^2[n]$$

$$\sigma_f^2 = 2\sigma_e^2 \left(\frac{1+p_1p_2}{1-p_1p_2}\right)\frac{1}{(1-p_1^2)(1-p_2^2)}$$

Choosing for example : $p_1 = \frac{1}{3}$ $p_2 = \frac{1}{5}$

$$\sigma_f^2 = 2\sigma_e^2 \left(\frac{1+1/15}{1-1/15}\right)\frac{1}{(1-\frac{1}{9})(1-\frac{1}{25})}$$

$$\sigma_f^2 = 2\sigma_e^2 \left(\frac{16}{14}\right)\frac{1}{\frac{8}{9}\cdot\frac{24}{25}}$$

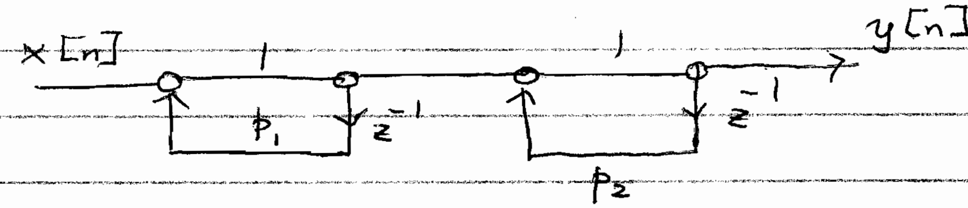
$$\sigma_f^2 = 2\sigma_e^2 \left(\frac{\cancel{8}}{7}\right)\frac{\overset{3}{\cancel{9}}\cdot 25}{\cancel{8}\cdot \cancel{24}\,8}$$

$$\sigma_f^2 = 2\sigma_e^2 \frac{75}{56} = \frac{150}{56}\sigma_e^2$$

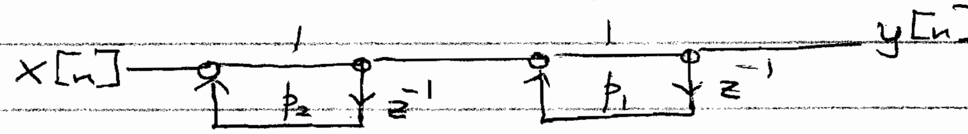
$$= \frac{75}{28}\sigma_e^2 = 2.6785\,\sigma_e^2$$

Cascade Form:

S_1



S_2



$$S_1 : \quad \sigma_f^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2[n] + \frac{\sigma_e^2}{1-p_2^2}$$

$$S_2 : \quad \sigma_f^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2[n] + \frac{\sigma_e^2}{1-p_1^2}$$

$$\sigma_f^2 = \begin{cases} S_1: \dfrac{\sigma_e^2}{1-p_2^2} + \sigma_e^2 \left(\dfrac{1+p_1p_2}{1-p_1p_2}\right) \dfrac{1}{(1-p_1^2)(1-p_2^2)} \\ S_2: \dfrac{\sigma_e^2}{1-p_1^2} + \sigma_e^2 \left(\dfrac{1+p_1p_2}{1-p_1p_2}\right) \dfrac{1}{(1-p_1^2)(1-p_2^2)} \end{cases}$$

For $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{5}$

$$\sigma_f^2 = \begin{cases} \dfrac{\sigma_e^2}{1-\frac{1}{25}} + \sigma_e^2 \left(\dfrac{1+1/15}{1-1/15}\right) \dfrac{1}{\frac{8}{9}\,\frac{24}{25}} & S_1 \\ \dfrac{\sigma_e^2}{1-\frac{1}{9}} + \sigma_e^2 \left(\dfrac{16}{14}\right) \dfrac{25 \cdot 9}{8 \cdot 24} & S_2 \end{cases}$$

$$\sigma_f^2 = \begin{cases} \left(\dfrac{25}{24} + \dfrac{16}{14} \cdot \dfrac{25 \cdot 9}{8 \cdot 24}\right)\sigma_e^2 & S_1 \\ \left(\dfrac{9}{8} + \dfrac{16}{14} \cdot \dfrac{25 \cdot 9}{8 \cdot 24}\right)\sigma_e^2 & S_2 \end{cases}$$

$$\sigma_f^2 = \begin{cases} \left(\dfrac{25}{24} + \dfrac{75}{56}\right)\sigma_e^2 & S_1 \\ \left(\dfrac{9}{8} + \dfrac{75}{56}\right)\sigma_e^2 & S_2 \end{cases}$$

$$\sigma_f^2 = \begin{cases} 2.3808\ \sigma_e^2 & S_1 \\ 2.4928\ \sigma_e^2 & S_2 \end{cases}$$

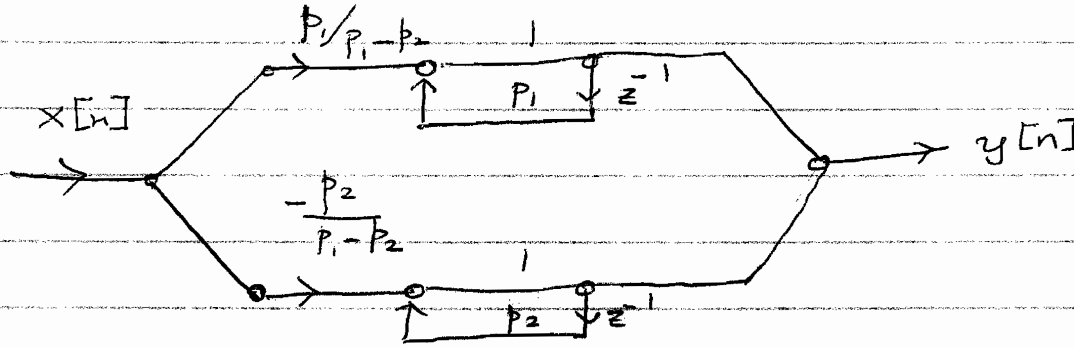
1.33

$56\overline{)75}$

$63 + 75 = 138$

$\frac{138}{56} = \frac{69}{28}$

Parallel Form:



$$\frac{p_1}{p_1-p_2} = \frac{\frac{1}{3}}{\frac{1}{3}-\frac{1}{5}} = \frac{\frac{1}{3}}{\frac{2}{15}}$$

$$= \frac{1}{3}\,\frac{15}{2} = 2.5$$

$$\frac{-p_2}{p_1-p_2} = \frac{-\frac{1}{5}}{\frac{2}{15}} = -1.5$$

$$\sigma_f^2 = \frac{2\sigma_e^2}{1-p_1^2} + \frac{2\sigma_e^2}{1-p_2^2}$$

$$= 2\sigma_e^2\left[\frac{1}{1-\frac{1}{9}} + \frac{1}{1-\frac{1}{25}}\right]$$

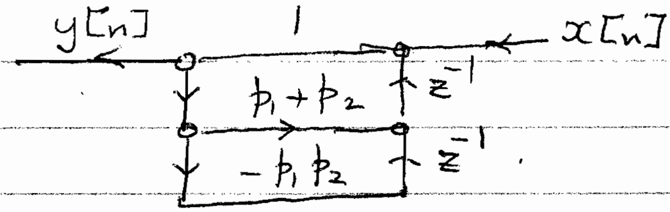
$$= 2\sigma_e^2\left[\frac{9}{8} + \frac{25}{24}\right]$$

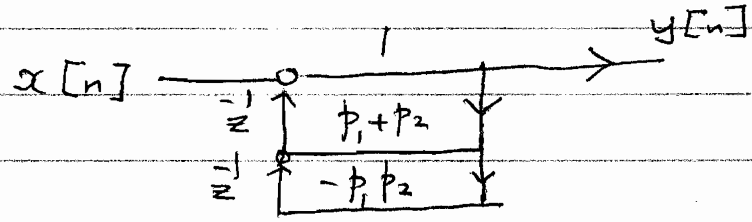
$$= 2\sigma_e^2\left[1.0416 + 1.125\right]$$

$$= 4.33\,\sigma_e^2$$

$$\begin{array}{r} 1.1250 \\ 1.0416 \\ \hline 2.1666 \\ \hline 4.3332 \end{array}$$

Transposed Direct Form:





Same as direct form:

Structure	σ_f^2
Direct	$2.6785\ \sigma_e^2$
S_1	$2.3808\ \sigma_e^2$
S_2	$2.4928\ \sigma_e^2$
Parallel	$4.332\ \sigma_e^2$
Transpose	$2.6785\ \sigma_e^2$

For this example: S_1 produces least noise