Convolution Based Algorithms

Goertzel Algorithm

Periodicity of Twiddle factors:

$$W_N^{-kN} = e^{-j\frac{2\pi}{N}kN} = W_N^{N} = 1$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}, \quad 0 \le k \le N-1$$

$$X[k] = W_N^{-kN} \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{-k(N-n)}$$

Define

$$y_k[n] = \sum_{r=-\infty}^{\infty} x[r]\, W_N^{-k(n-r)}\, u[n-r]$$

$$= x[n] * W_N^{-kn}\, u[n]$$

Since $x[n]$ has support over $0 \le n \le N-1$

$$y_k[n] = \sum_{r=0}^{N-1} x[r]\, W_N^{-k(n-r)}\, u[n-r]$$

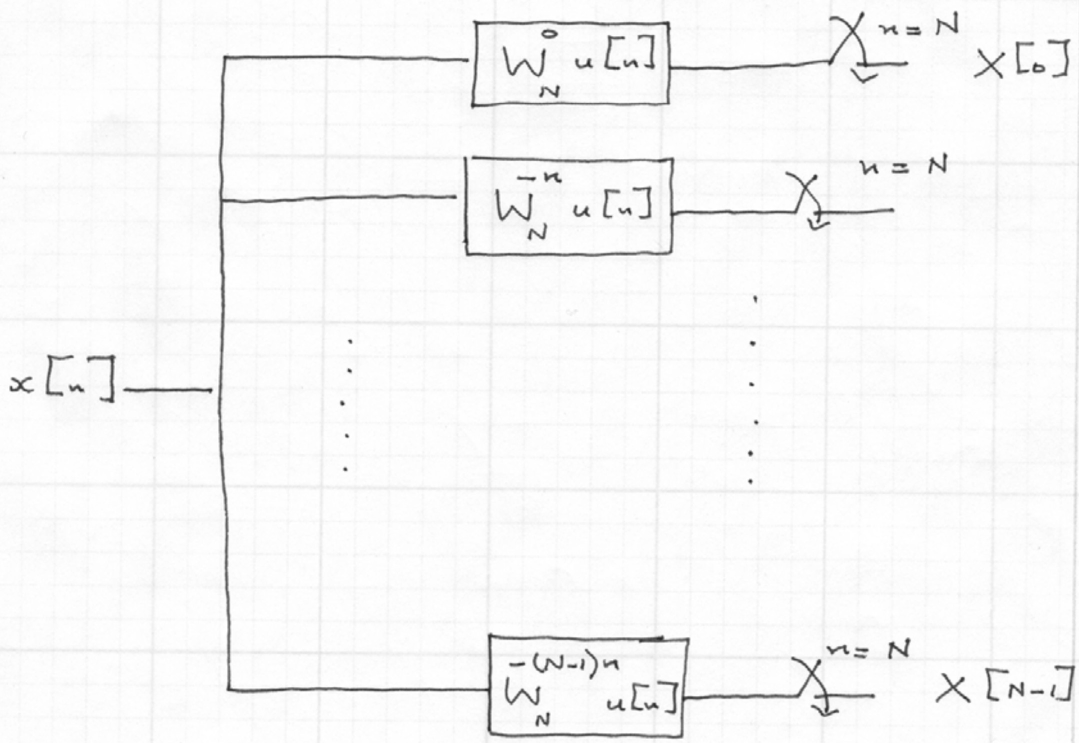
$$= \sum_{r=0}^{N-1} x[r]\, W_N^{-k(n-r)}$$

Relating $y_k[n]$ we can see that

$$y_k[N] = \sum_{r=0}^{N-1} x[r]\, W_N^{-k(N-r)}$$

$$= X[k]$$

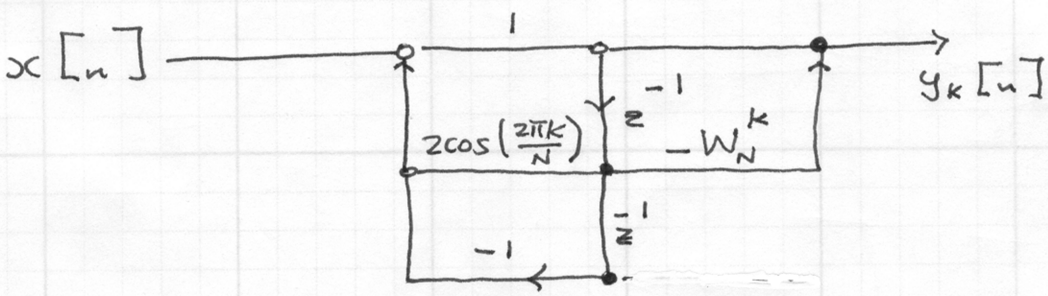
This suggests the following filter bank formulation for the DFT



Filter Implementation

$$H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}}$$

$$H_k(z) = \frac{1 - W_N^{k} z^{-1}}{1 - 2\cos\left(\frac{2\pi}{N}k\right) z^{-1} + z^{-2}}$$



- Number of real multiplies

  $\sim N^2$

- Number of real adds

  $\sim 2N^2$

Chirp Z Transform

Goal: To evaluate $X(e^{j\omega})$ at

$$\omega_k = \omega_0 + k\,\Delta\omega\,, \quad 0 \le k \le M-1$$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega_k n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\omega_0 n}\, e^{-jk\Delta\omega n}$$

Define $e^{-j\Delta\omega} = W$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega_0 n}\, W^{nk}$$

$$nk = \frac{1}{2}\left\{ n^2 + k^2 - (k-n)^2 \right\}$$

substituting this identity we have

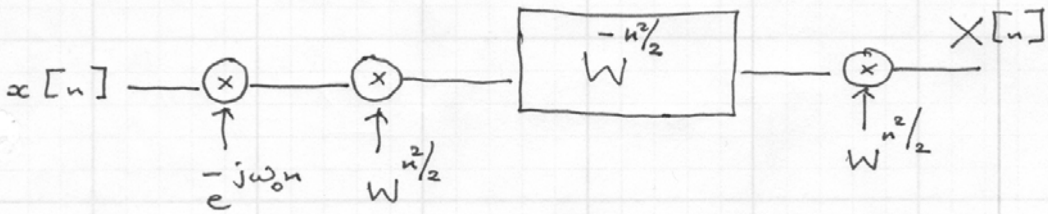
$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega_0 n}\, W^{n^2/2}\, W^{-\frac{1}{2}(k-n)^2} \quad W^{k^2/2}$$

define $g[n] = x[n]\, e^{-j\omega_0 n}\, W^{n^2/2}$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} g[n]\, W^{-1/2\,(k-n)^2}\, W^{k^2/2}$$

$$X(e^{j\omega_k}) = \left(g[k] * W^{-k^2/2} \right) W^{k^2/2}$$

Block Diagram:



Using the commutative property of convolution

$$X(e^{j\omega_k}) = \sum_{-\infty}^{\infty} W^{-n^2/2}\, g[k-n] \quad W^{k^2/2}$$

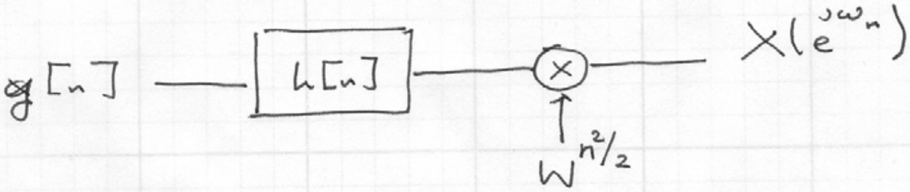
- $0 \le -n + k \le N-1$
- $0 \le k \le M-1$

$$\Rightarrow \quad -(N-1) \le n \le M-1$$

$$X(e^{j\omega_k}) = \sum_{n=-(N-1)}^{M-1} W^{-n^2/2}\, g[k-n]\; W^{k^2/2}$$

$$X(e^{j\omega_n}) = \left(h[n] * g[n]\right) W^{n^2/2},$$

where $h[n] = \begin{cases} W^{-n^2/2}, & -(N-1) \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$



Finite Convolution Model

- $\Delta\omega = \frac{2\pi}{N}$, $\omega_0 = 0$
  we get the DFT
- Allows sampling of $X(e^{j\omega})$
  around circle of radius ω_0
- Useful when processor optimized
  for implementing $*$ available