Cooley - Tukey Algorithm :

Composite Factor Algorithm:

$$N = N_1 N_2 , \quad N_1 \;\&\; N_2 \neq 1$$

Index Mapping :

$$\begin{aligned} n &= N_2 n_1 + n_2 , & 0 \le n_2 \le N_2 - 1 \\ & & 0 \le n_1 \le N_1 - 1 \\ k &= k_1 + N_1 k_2 , & 0 \le k_1 \le N_1 - 1 \\ & & 0 \le k_2 \le N_2 - 1 \end{aligned}$$

- Transformation from (n_1, n_2) pair to n unique
- All possible combinations accounted for in map

$$X\left[((k_1 + N_1 k_2))_N\right] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[N_2 n_1 + n_2]\, W_N^{(N_2 n_1 + n_2)(k_1 + N_1 k_2)}$$

$$X\left[((k_1 + N_1 k_2))_N\right] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[N_2 n_1 + n_2]\, W_N^{N_2 n_1 k_1}\, W_N^{n_2 k_1}\, W_N^{N_1 n_2 k_2}$$

$$= \underbrace{\sum_{n_2=0}^{N_2-1} \underbrace{\sum_{n_1=0}^{N_1-1} x[N_2 n_1 + n_2]\, W_{N_1}^{n_1 k_1}}_{N_2 \;\; N_1\text{-pt DFT's}} \; \underbrace{W_N^{n_2 k_1}}_{\text{Twiddle factors}} \; W_{N_2}^{n_2 k_2}}_{N_1 \;\; N_2\text{-pt DFT's}}$$

(1) $$g[n_2, k_1] \triangleq \sum_{n_1=0}^{N_1-1} x[N_2 n_1 + n_2]\, W_{N_1}^{n_1 k_1}$$

(2) $$\tilde{g}[n_2, k_1] \triangleq g[n_2, k_1]\, W_N^{n_2 k_1}$$

(3) $$X[k_1, k_2] \triangleq \sum_{n_2=0}^{N_2-1} \tilde{g}[n_2, k_1]\, W_{N_2}^{n_2 k_2}$$

(4) $$X[k] = X[((k_1 + N_1 k_2))_N]$$

Example:

$N = 3 . 6 = 18$ (composite Factor)

$n = 6 n_1 + n_2$, $0 \le n_1 \le 2$
$0 \le n_2 \le 5$

n_1 \ n_2	0	1	2	3	4	5
0	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$
1	$x[6]$	$x[7]$	$x[8]$	$x[9]$	$x[10]$	$x[11]$
2	$x[12]$	$x[13]$	$x[14]$	$x[15]$	$x[16]$	$x[17]$

6 3pt DFT's

k_1 \ n_2	0	1	2	3	4	5
0	$g[0,0]$	$g[0,1]$	$g[0,2]$	$g[0,3]$	$g[0,4]$	$g[0,5]$
1	$g[1,0]$	$g[1,1]$	$g[1,2]$	$g[1,3]$	$g[1,4]$	$g[1,5]$
2	$g[2,0]$	$g[2,1]$	$g[2,2]$	$g[2,3]$	$g[2,4]$	$g[2,5]$

$$\tilde{g}[n_2,k_1] = g[n_2,k_1]\, W_{18}^{n_2 k_1}$$

$k_1 \backslash n_2$	0	1	2	3	4	5
0	$\tilde{g}[0,0]$	$\tilde{g}[0,1]$	$\tilde{g}[0,2]$	$\tilde{g}[0,3]$	$\tilde{g}[0,4]$	$\tilde{g}[0,5]$
1	$\tilde{g}[1,0]$	$\tilde{g}[1,1]$	$\tilde{g}[1,2]$	$\tilde{g}[1,3]$	$\tilde{g}[1,4]$	$\tilde{g}[1,5]$
2	$\tilde{g}[2,0]$	$\tilde{g}[2,1]$	$\tilde{g}[2,2]$	$\tilde{g}[2,3]$	$\tilde{g}[2,4]$	$\tilde{g}[2,5]$

$\downarrow$ Transform along rows using 3 6pt DFT's

$k_1 \backslash k_2$	0	1	2	3	4	5
0	$X[0]$	$X[3]$	$X[6]$	$X[9]$	$X[12]$	$X[15]$
1	$X[1]$	$X[4]$	$X[7]$	$X[10]$	$X[13]$	$X[16]$
2	$X[2]$	$X[5]$	$X[8]$	$X[11]$	$X[14]$	$X[17]$

Prime Factor Algorithm:

$$N = N_1 N_2, \qquad \gcd(N_1, N_2) = 1$$

Index Mapping:

$$n = ((A n_1 + B n_2))_N, \qquad 0 \le n_1 \le N_1 - 1$$
$$0 \le n_2 \le N_2 - 1$$

$$k = ((C k_1 + D k_2))_N, \qquad 0 \le k_1 \le N_1 - 1,$$
$$0 \le k_2 \le N_2 - 1$$

Particular choice:

$$A = N_2, \quad B = N_1, \quad C = N_2 ((N_2^{-1}))_{N_1}$$
$$D = N_1 ((N_1^{-1}))_{N_2}$$

makes twiddle factors disappear, i.e.,

$$W_N^{(An_1 + Bn_2)(Ck_1 + Dk_2)}$$

$$= W_N^{AC\, n_1 k_1} \; W_N^{AD\, n_1 k_2} \; W_N^{BC\, n_2 k_1} \; W_N^{BD\, n_2 k_2}$$

$$= W_N^{N_2 n_1 k_1} \; \cancelto{1}{W_N^{N_1 N_2 ((N_1^{-1}))_{N_2}}} \; \cancelto{1}{W_N^{N_1 N_2 ((N_2^{-1}))_{N_1}}} \; W_N^{N_1 n_2 k_2}$$

$$= W_{N_1}^{n_1 k_1} \; W_{N_2}^{n_2 k_2}$$

- $((N_1^{-1}))_{N_2}$ denotes multiplicative inverse of N_1 modulo N_2

$$X[((Ck_1 + Dk_2))_N]$$

$$= \underbrace{\sum_{n_2=0}^{N_2-1} \underbrace{\sum_{n_1=0}^{N_1-1} x[N_2 n_1 + N_1 n_2]\, W_{N_1}^{n_1 k_1}}_{N_2 \quad N_1\text{-pt DFT's}} \; W_{N_2}^{n_2 k_2}}_{N_1 \quad N_2\text{-pt DFT's}}$$

Example

$$N = 12 = \underbrace{3}_{N_1} \cdot \underbrace{4}_{N_2}$$

$\gcd(N_1, N_2) = 1$, $\quad ((N_1^{-1}))_{N_2} = 3$

$((N_2^{-1}))_{N_1} = 1$

$3x = 4p + 1$

$p = 2,\ x = 3$

Index mapping :

$$n = ((4n_1 + 3n_2))_{12}$$

$$k = ((4k_1 + 9k_2))_{12}$$

$4x = 3q + 1$

$q =$ [illegible], $x = 1$

$$W_N^{nk} = W_N^{(4n_1 + 3n_2)(4k_1 + 9k_2)}$$

$$W_N^{nk} = W_{12}^{16 n_1 k_1} \; \cancelto{1}{W_{12}^{36 n_1 k_2}} \cdot \cancelto{1}{W_{12}^{12 n_2 k_1}} \; W_{12}^{27 n_2 k_2}$$

$$W_N^{nk} = W_{12}^{4 n_1 k_1} \; W_{12}^{3 n_2 k_2}$$

$$= W_3^{n_1 k_1} \; W_4^{n_2 k_2}$$

Input ordering:

n_1 \ n_2	0	1	2	3
0	$x[0]$	$x[3]$	$x[6]$	$x[9]$
1	$x[4]$	$x[7]$	$x[10]$	$x[1]$
2	$x[8]$	$x[11]$	$x[2]$	$x[5]$

4 3pt DFT's

k_1 \ n_2	0	1	2	3
0	$G[0,0]$	$G[0,1]$	$G[0,2]$	$G[0,3]$
1	$G[1,0]$	$G[1,1]$	$G[1,2]$	$G[1,3]$
2	$G[2,0]$	$G[2,1]$	$G[2,2]$	$G[2,3]$

Transform along rows

3 4pt DFT's

Output Ordering

k_1 \ k_2	0	1	2	3
0	$X[0]$	$X[9]$	$X[6]$	$X[3]$
1	$X[4]$	$X[1]$	$X[10]$	$X[7]$
2	$X[8]$	$X[5]$	$X[2]$	$X[11]$

Computational Complexity:

$$N = N_1 N_2$$

$$\mu(N) = N_2 \mu(N_1) + N_1 \mu(N_2) + \underbrace{N}_{\text{Twiddle Factors}}$$

$$N = N_1 N_2 N_3$$

$$\begin{aligned}
\mu(N) &= N_2 N_3 \mu(N_1) + N_1 \mu(N_2 N_3) + N. \\
&= N_2 N_3 \mu(N_1) + N_1 N_2 \mu(N_3) \\
&\quad + N_1 N_3 \mu(N_2) + N_1 N_2 N_3 \\
&\quad + N
\end{aligned}$$

$$\mu(N) = N \left(\frac{\mu(N_1)}{N_1} + \frac{\mu(N_2)}{N_2} + \frac{\mu(N_3)}{N_3} + 2 \right)$$

$$< N^2 \quad \text{because}$$

$$\frac{\mu(N_1)}{N_1} + \frac{\mu(N_2)}{N_2} + \frac{\mu(N_3)}{N_3} < N$$