Cascade of Decimation Systems

In class, we looked at the decimation operation in detail and observed that it was a spectral zoom and magnify operation. In this exercise, we will look at the effect of cascading two binary decimation modules. In the time domain this cascade operation can be described as:

$$f[n] = \sum_{k=-\infty}^{\infty} h_{\text{lp}}[k] y[2n-k],$$

where $y[n]$ is the output of the first decimation module. This intermediate output signal in turn can be related back to the input as:

$$y[n] = \sum_{k=-\infty}^{\infty} h_{\text{lp}}[k] x[2n-k]$$

Substituting this expression into the earlier one we obtain:

$$f[n] = \sum_{p=-\infty}^{\infty} h_{\text{lp}}[p] \sum_{k=-\infty}^{\infty} h_{\text{lp}}[p] x[4n-2k-p].$$

The corresponding result after application of the Noble identities is given by

$$f[n] = \sum_{k=-\infty} x[k] h_{\text{eff}}[4n-k] = \sum_{k=-\infty}^{\infty} h_{\text{eff}}[k] x[4n-k],$$

where $h_{\text{eff}}[n]$ is the impulse response of the system described by:

$$H_{\text{eff}}(z) = H_{\text{lp}}(z) H_{\text{lp}}(z^2),$$

where $H_{\text{lp}}(z)$ is a lowpass filter with cut-off frequency $\omega_c = \frac{\pi}{2}$ and unity gain in the pass-band. Note that: (a) the filter $H_{\text{eff}}(z)$ is in effect a lowpass filter with a cut-off frequency $\omega_c = \frac{\pi}{4}$, (b) the effective downsampling factor is $M_{\text{eff}} = 4$, and that the imaging terms resulting from the $H_{\text{lp}}(z^2)$ term are eliminated by the term $H_{\text{lp}}(z)$. If we now cascade K binary decimation units then we obtain an effective decimation system of the form:

$$H_{\text{eff}}(z) = \prod_{k=1}^{K-1} H_{\text{lp}}(z^k), \quad M_{\text{eff}} = 2^K.$$

The effective system would represent a spectral zoom operation into the region $|\omega| \leq \frac{\pi}{2^K}$. Using a suitable combination of lowpass and high-pass filters we could in a similar fashion zoom into the other regions of the frequency-domain. In our discussion of *multi-resolution analysis* or wavelets, this operation of zooming into a desired region in the frequency domain will be termed as *scale-space analysis*.