Upsampling and Downsampling

In the previous section we looked at upsampling and the downsampling as specific forms of sampling. In this section, we will look at these operations from a matrix framework. Consider a signal $x[n]$, obtained from Nyquist sampling of a bandlimited signal, of length L.

Downsampling operation

Consider the downsampling operation by a factor of M, given by:

$$y[n] = \downarrow M(x[n]) = x[Mn],$$

This relation can be expressed in finite dimensions as the matrix equation:

$$\mathbf{y} \; = \; \mathbf{D}_{P,M}\mathbf{x}.$$

where we define the following vectors:

$$\mathbf{x} = \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[L-1] \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y[0] \\ y[1] \\ \vdots \\ y[P-1] \end{pmatrix}, \quad P = \text{floor}\left(\frac{L}{M}\right).$$

and the downsampling matrix $\mathbf{D}_{P,M}$ by:

$$\mathbf{D}_{P,M} = \begin{pmatrix} 1 & & & \mathbf{0}_{L-1} \\ \mathbf{0}_M & 1 & & \mathbf{0}_{L-M-1} \\ \mathbf{0}_{2M} & & 1 & \mathbf{0}_{L-2M-1} \\ \vdots & & \vdots & \vdots \end{pmatrix}.$$

The (i,j)-th element of the downsampling matrix $\mathbf{D}_{P,M}$ is given by

$$\{\mathbf{D}_{P,M}\}_{ij} = \downarrow M(\delta[i-j]) = \delta[Mi-j].$$

Upsampling operation

Consider the upsampling operation by a factor of R given by the relation:

$$y[n] \; = \uparrow R(x[n]) \; = \; x\left[\frac{n}{R}\right].$$

We can formulate the above system in finite dimensions as a matrix equation via:

$$\mathbf{y} = \mathbf{U}_{LR,L}\mathbf{x} = \mathbf{U}_{K,L}\mathbf{x},$$

where we define the vector $\mathbf{y}$ and the upsampling matrix $\mathbf{U}_{K,L}$ as follows:

$$\mathbf{y} = \begin{pmatrix} y[0] \\ y[1] \\ \vdots \\ y[K-1] \end{pmatrix}, \quad \mathbf{U}_{K,L} = \begin{pmatrix} 1 & & \mathbf{0}_{L-1} \\ & \mathbf{0}_{(R-1)\times L} & \\ 0 & 1 & \mathbf{0}_{L-2} \\ & \mathbf{0}_{(R-1)\times L} & \\ \mathbf{0}_2 & 1 & \mathbf{0}_{L-3} \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

The (i,j)-th element of the upsampling matrix $\mathbf{U}_{K,L}$ is given by:

$$\{\mathbf{U}_{K,L}\}_{ij} = \uparrow R(\delta[i-j]) = \delta\left[\frac{i}{R} - j\right] = \delta[i-Rj].$$

Properties of the Multirate Matrices

Orthogonality

Specifically note that columns of the upsampling matrix and the rows of the downsampling matrix are orthogonal, i.e.,

$$\mathbf{D}_{L,P}\mathbf{D}_{P,L}^T = \mathbf{I}_{L,L}, \quad \mathbf{U}_{L,K}^T\mathbf{U}_{K,L} = \mathbf{I}_{L,L}.$$

Also note that the upsampling matrix is the transpose of the downsampling matrix, i.e.,

$$\mathbf{U}_{L,P} = \mathbf{D}_{P,L}^T.$$

Existence of Inverse

Note further that the upsampling matrix is the right-inverse of the downsampling matrix, i.e.,

$$\mathbf{D}_{L,P}\mathbf{U}_{P,L} = \mathbf{I}_{L,L}.$$

Note that the matrices $\mathbf{U}_{P,L}$ and $\mathbf{U}_{L,P}$ do not commute, i.e.,

$$\mathbf{D}_{L,K}\mathbf{U}_{K,L} \neq \mathbf{U}_{K,L}\mathbf{D}_{L,K}.$$

In fact the symmetric matrix $\mathbf{P} = \mathbf{U}_{L,P}\mathbf{D}_{P,L} = \mathbf{D}_{L,P}^T\mathbf{D}_{P,L}$ is a projection matrix onto the space of signal samples over the set of smaller sampling indices $n = 0, R, 2R, \ldots L-1$ as is evident from:

$$\mathbf{P}^2 = \mathbf{D}_{L,P}^T\mathbf{D}_{P,L}\mathbf{D}_{L,P}^T\mathbf{D}_{P,L} = \mathbf{D}_{L,P}^T\mathbf{D}_{P,L} = \mathbf{P}.$$

Commutation

We also saw in class that the matrices $\mathbf{D}_M$ and $\mathbf{U}_R$ will commute when M and R are mutually coprime, i.e.,

$$\mathbf{D}_{PR,K}\mathbf{U}_{K,L} = \mathbf{U}_{PR,P},\mathbf{D}_{P,L}, \quad \gcd(R, M) = 1$$

Matrix Norm

The ranks of the matrices $\mathbf{U}$ and $\mathbf{D}$ are given by:

$$\mathrm{rank}(\mathbf{U}_{K,L}) = L, \quad \mathrm{rank}(\mathbf{D}_{P,L}) = P.$$

Note further that all the singular values of $\mathbf{U}_{K,L}$ and $\mathbf{D}_{P,L}$ are unity, i.e.,

$$\mathrm{eig}(\mathbf{D}_{P,L}\mathbf{D}_{L,P}^T) = \mathrm{diag}(\mathrm{ones}(1, P)), \ \mathrm{eig}(\mathbf{U}_{L,K}^T\mathbf{U}_{K,L}) = \mathrm{diag}(\mathrm{ones}(1, L))$$

and consequently the spectral norm of either matrix is unity, i.e.,

$$\mathrm{norm}(\mathbf{D}_{P,L}) = \mathrm{norm}(\mathbf{U}_{K,L}) = 1.$$

Boundedness

Ofcourse this is just a reflection of the fact that the upsampling and downsampling operations are BIBO linear operations, i.e.,

$$||\mathbf{D}_{P,L}\mathbf{x}||_2 < ||\mathbf{x}||_2, \quad ||\mathbf{D}_{K,L}\mathbf{x}|| = ||\mathbf{x}||_2$$