DISCRETE FOURIER SERIES

Consider a N pt sequence $x[n]$ and its periodically extended version $\tilde{x}[n]$:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN]$$

$$x[n] = \begin{cases} \tilde{x}[n] , & 0 \le n \le N-1 \\ 0 , & \text{otherwise} \end{cases}$$

The periodically extended version has a Fourier series representation:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \, e^{j\frac{2\pi}{N}nk}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] \, e^{-j\frac{2\pi}{N}nk}$$

Employing the notation

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \, W_N^{-nk}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] \, W_N^{nk}$$

Note that $\tilde{x}[n] = \tilde{x}[n+N]$

$\tilde{X}[k] = \tilde{X}[k+N]$

Using the Notation:

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

The DFS pair can be formulated as

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

(Using Notation)

$$\tilde{x}[n] = \sum_{n=0}^{N-1} X[k]\, W_N^{-nk}$$

The DFT pair is then defined via:

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n]\, W_N^{nk}, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \sum_{k=0}^{N-1} X[k]\, W_N^{-nk}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\left.\begin{aligned} \tilde{X}[k] &= X[((k))_N] \\ \tilde{x}[n] &= x[((n))_N] \end{aligned}\right\} \text{Modulo Notation}$$

Frequency Sampling:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad (\text{DTFT})$$

If $x[n]$ is a N point sequence

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

The DTFT of this sequence sampled at $\omega_k = \frac{2\pi}{N} \cdot k$ is

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N} nk}$$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

$$= X[k] \quad (\text{DFT})$$

The DFT is just a frequency domain sampled version of the DTFT for finite length signals

Relation between $X(e^{j\omega})$ & $X[k]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n}$$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\; \frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n}$$

$$\Phi_N[k,\omega] = \frac{1}{N}\left(\frac{1 - e^{-j(\omega - \frac{2\pi}{N}k)N}}{1 - e^{-j(\omega - \frac{2\pi}{N}k)}}\right)$$

$$\Phi_N[k,\omega] = \frac{1}{N}\left(\frac{\sin\left(\left(\frac{N}{2}\right)\left(\omega - \frac{2\pi}{N}k\right)\right)}{\sin\left(\omega - \frac{2\pi}{N}k\right)}\right) e^{-j\left(\omega - k\frac{2\pi}{N}\right)\left(\frac{N-1}{2}\right)}$$

$$D_N(\omega) = \frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\Phi_N[k,\omega] = \frac{1}{N}\, D_N\left(\omega - \frac{2\pi}{N}k\right) e^{-j\left(\omega - \frac{2\pi}{N}k\right)\left(\frac{N-1}{2}\right)}$$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\; \Phi_N[k,\omega]$$

(Dirichlet Interpolation formula)

Characterestics of $\Phi_N(\omega, k]$

$$\Phi_N[k, \omega) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n}$$

$$\Phi_N\left[k, \frac{2\pi}{N}r\right] = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(r-k)n}$$

$$= \frac{1}{N} \left\langle e^{-j\frac{2\pi}{N}rn}, e^{-j\frac{2\pi}{N}kn} \right\rangle$$

$$\Phi_N\left[k, \frac{2\pi}{N}r\right] = \begin{cases} 1, & r = k \\ 0, & \text{otherwise} \end{cases} \qquad (1)$$

$$\Phi_N[0, 0] = 1 \qquad (2)$$

$$|\Phi_N[k, \omega)| \leq \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1 \qquad \forall\, \omega \in [-\pi, \pi], \quad 0 \leq k \leq N-1 \qquad (3)$$

$\Rightarrow$ Maximum value of interpolating function is 1

$$\Phi_N[k, 0] = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk} = \delta[((k))_N] \qquad (4)$$

$\Rightarrow$ $\Phi_N(\omega, k]$ is a true interpolating function

Time Aliasing:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-nk}$$

$$\tilde{x}[n] = \sum_{p=-\infty}^{\infty} x[n-pN]$$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Sampling of the DTFT in the time-domain corresponds to time aliasing in the time-domain

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{p=0}^{N-1} x[p]\, W_N^{kp}\, W_N^{-nk}$$

$$= \sum_{p=0}^{N-1} x[p]\, \frac{1}{N}\sum_{k=0}^{N-1} W_N^{(p-n)k}$$

$$= \sum_{p=0}^{N-1} x[p]\, \delta[((p-n))_N],$$

where $\frac{1}{N}\sum_{k=0}^{N-1} W_N^{(p-n)k} = \delta[((p-n))_N]$

$$\tilde{x}[n] = \sum_{r=-\infty}^{+\infty} x[n+rN]$$