Properties of the DFT

Circular Shift

$$X_1[k] = e^{-j\frac{2\pi}{N}mk} X[k] \, , \quad 0 \le k \le N-1$$

$$\tilde{x}_1[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_1[k] \, W_N^{-nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{mk} \, W_N^{-nk}$$

$$\tilde{x}_1[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{-(n-m)k}$$

$$= x[((n-m))_N] = x[n-m+rN]$$

where $r \in I$

$$x[((n-m))_N] \overset{W}{\rightleftharpoons} e^{-j\frac{2\pi}{N}mk} X[k]$$

$$x_1[n] = \begin{cases} x[((n-m))_N] \, , & 0 \le n \le N-1 \\ 0 \, , & \text{otherwise} \end{cases}$$

Circular Convolution:

$$X_1[k]\,X_2[k] \xrightarrow{W^{-1}} \;?$$

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k]\,W_N^{-nk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]\,X_2[k]\,W_N^{-nk}$$

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k] \sum_{p=0}^{N-1} x_2[p]\,W_N^{kp}\;W_N^{-nk}$$

$$= \frac{1}{N}\sum_{p=0}^{N-1} x_2[p] \sum_{k=0}^{N-1} X_1[k]\,W_N^{-k(n-p)}$$

$$= \sum_{p=0}^{N-1} x_2[p]\;x_1[((n-p))_N]$$

$$= \sum_{p=0}^{N-1} x_2[p]\;x_1[n-p+\ell N], \quad \text{where}$$

$$\ell \in I \quad \& \quad n-p+\ell N \in \{0,1,N-1\}$$

$$x_3[n] = x_1[n] \; Ⓝ \; x_2[n]$$

$$= x_2[n] \; Ⓝ \; x_1[n]$$

Conjugation:

$$x^*[n] \xrightarrow{W} \sum_{n=0}^{N-1} x^*[n]\, W_N^{nk}$$

$$= \left[\sum_{n=0}^{N-1} x[n]\, W_N^{n(-k)} \right]^*$$

$$= X^*[((-k))_N]$$

$$X^*[k] \xrightarrow{W^{-1}} \frac{1}{N} \sum_{k=0}^{N-1} X^*[k]\, W_N^{-nk}$$

$$= \left(\frac{1}{N} \sum_{k} X[k]\, W_N^{k(-n)} \right)^*$$

$$= x^*[((-n))_N]$$

Definitions

For general complex sequences

$$\text{Real}\{x[n]\} \triangleq \frac{x[n] + x^*[n]}{2}$$

$$\text{Imag}\{x[n]\} \triangleq \frac{x[n] - x^*[n]}{2j}$$

$$\text{CSY}\{x[n]\} \triangleq \frac{x[n] + x^*[-n]}{2}$$

$$\text{CASY}\{x[n]\} \triangleq \frac{x[n] - x^*[-n]}{2}$$

Symmetry Properties

$$\text{Real}\{x[n]\} = \frac{x[n] + x^*[n]}{2}$$

$$\text{DFT}\left\{\frac{x[n] + x^*[n]}{2}\right\}$$

$$= \frac{1}{2} X[((k))_N] + \frac{1}{2} X^*[((-k))_N]$$

$$= \frac{1}{2}\left\{X[((k))_N] + X^*[((-k))_N]\right\}, \quad 0 \le k \le N-1$$

If $x_{ep}[n]$ is defined via:

$$x_{ep}[n] = \frac{1}{2}\left\{x[n] + x^*[((-n))_N]\right\}$$

$$, 0 \le n \le N-1$$

$$\text{DFT}\{\text{Real}(x[n])\} \equiv X_{ep}[k]$$

$$= \text{CSY}(X[k])$$

$$\text{DFT}\{j\,\text{Imag}(x[n])\} = X_{op}[k]$$

$$= \text{CASY}(X[k])$$

Matrix Transform Interpretation

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{+nk}$$

$$X[0] = x[0]\, W_N^{00} + x[1]\, W_N^{10} + \dots\; x[N-1]\, W_N^{N-1\,0}$$

$$X[1] = x[0]\, W_N^{01} + x[1]\, W_N^{11} + \dots\dots$$

$$\vdots$$

$$X[N-1] = x[0]\, W_N^{0\,N-1} + x[1]\, W_N^{1\,N-1} + \dots\; x[N-1]\, W_N^{N-1\,N-1}$$

$$\underbrace{\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix}}_{\underline{X}} = \underbrace{\begin{pmatrix} W_N^{00} & W_N^{10} & \dots\dots & W_N^{N-1\,0} \\ \vdots & & & \\ W_N^{0\,N-1} & W_N^{1\,N-1} & \dots\dots & W_N^{N-1\,N-1} \end{pmatrix}}_{W_{N\times N}} \underbrace{\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix}}_{\underline{x}}$$

$$W_N = e^{-j\frac{2\pi}{N}}, \quad W_N^{(0)} = 1$$

$$c_i = \begin{pmatrix} W_N^{i0} \\ W_N^{i1} \\ \vdots \\ W_N^{i\,N-1} \end{pmatrix} = \begin{pmatrix} 1 \\ W_N^{i} \\ W_N^{2i} \\ W_N^{(N-1)i} \end{pmatrix}, \quad 0 \le i \le N-1$$

$$\langle c_i, c_j \rangle = \sum_{p=0}^{N-1} W_N^{pi}\, W_N^{-pj}$$

$$= \sum_{p=0}^{N-1} W_N^{p(i-j)} = N\delta[((i-j))_N]$$

Characterestics

(a) Columns are orthogonal, i.e,

$$\langle c_i, c_j \rangle = 0 \quad , \quad i \neq j$$

(b) $W^T W = N I$

(c) Eigenvalues are $1, -1, j, -j$, $\lambda^4 = 1$

(d) Has built-in symmetries

(i) Vandermonte structure

(ii) Circulant

Built in structure will enable us to develop fast algorithms for its computation

Direct Computation:

For every DFT sample we require

- M complex multiplications & $M-1$ Complex adds

- $\mu(M)$ for all M samples

  M^2 multiplications & $M^2 - M$ adds

  $\approx M^2$ multiplies