Exponential Fourier Series

If the orthogonal countably infinite collection of functions $\{\phi_i(t), i = -\infty, \ldots, \infty\}$ forms a basis for the space $\mathbf{H}[a, b]$ then any function $f(t)$ in the square integrable space $\mathbf{H}[a, b]$. can be expressed as a unique linear combination of the members of this collection.

$$f(t) = \sum_{i=1}^{i=\infty} c_i \phi_i(t),$$

When the collection of functions $\{\phi_i(t), i = -\infty, \ldots, \infty\}$ is an orthogonal collection of functions these unique coefficients can be obtained by using the inner product $< f(t), \phi_k(t) >$:

$$\begin{aligned}
< f(t), \phi_k(t) > &= < \sum_{i=1}^{i=\infty} c_i \phi_i(t), \phi_k(t) > \\
< f(t), \phi_k(t) > &= \sum_{i=1}^{i=\infty} c_i < \phi_i(t), \phi_k(t) > \\
< f(t), \phi_k(t) > &= \sum_{i=1}^{i=\infty} c_i < \phi_k(t), \phi_k(t) > \delta_{i,k} \\
c_k &= \frac{< f(t), \phi_k(t) >}{< \phi_k(t), \phi_k(t) >},
\end{aligned}$$

where we have used the fact that the summation and the integration operators can be swapped because they are both linear operations and the integration variable and the summation index are independent of each other.

The synthesis and analysis relations for the function $f(t) \in \mathbf{H}[a, b]$ are given by

$$\begin{aligned}
f(t) &= \sum_{i=1}^{i=\infty} \left(\frac{< f(t), \phi_i(t) >}{< \phi_i(t), \phi_i(t) >} \right) \phi_i(t) \\
c_k &= \frac{1}{||\phi_k(t)||^2} \int_a^b f(\tau) \phi_k^*(\tau) d\tau
\end{aligned}$$

In the special case where the orthogonal collection of functions are the collection of complex exponentials:

$$\{\phi_i(t)\} = \exp\left(j \frac{2\pi}{T_o} it\right) \ , \ -\infty \leq i \leq \infty,$$

The synthesis and analysis relations can be rewritten to give the exponential Fourier series expansion for the function $f(t)$.

$$\begin{aligned}
f(t) &= \sum_{i=-\infty}^{i=\infty} c_i \exp\left(j \frac{2\pi}{T_o} it\right) \\
c_k &= \frac{1}{T_o} \int_0^{T_o} f(\tau) \exp\left(-j \frac{2\pi}{T_o} k\tau\right) d\tau.
\end{aligned}$$

The Fourier series coefficient for $k = 0$, c_0 is termed the dc coefficient or the average value of the function. The term $\omega_o = \frac{2\pi}{T_o}$ is called the fundamental frequency and integer multiples of ω_o are termed harmonics. The Fourier series expansion of the function $f(t)$ therefore can be treated as a spectral analysis and synthesis of the function $f(t)$.