Lattice Structures :

Consider a sequence of FIR filters with system functions

$$H_i(z) = A_i(z) \quad , \quad i = 0, 1, 2, 3, \ldots (M-1)$$

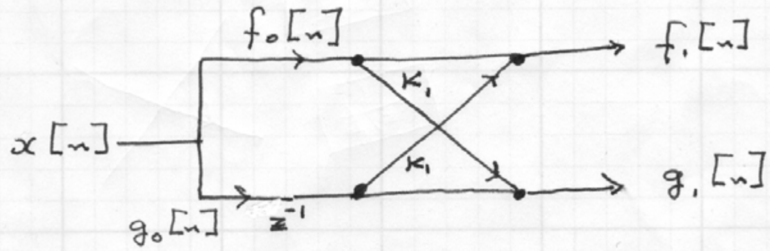
The polynomial $A_i(z)$ is defined via:

$$A_p(z) = 1 + \sum_{k=1}^{p} a_p[k] z^{-k}$$

Let $x[n]$ be the input to $A_p(z)$ and $y[n]$ be the output

$$y_p[n] = x[n] + \sum_{k=1}^{p} a_p[k]\, x[n-k]$$

First-order Lattice:

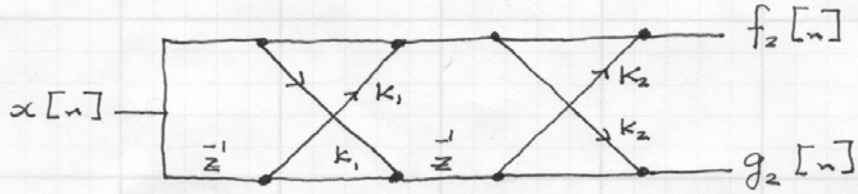


$$\left.\begin{aligned} f_1[n] &= f_0[n] + k_1\, g_0[n-1] \\ g_1[n] &= g_0[n-1] + k_1\, f_0[n] \end{aligned}\right\} \text{Time domain}$$

$$\frac{F_1(z)}{X(z)} = 1 + k_1 z^{-1} = 1 + a_1[1] z^{-1} = A_1(z)$$

$$\frac{G_1(z)}{X(z)} = z^{-1} + k_1 = z^{-1}(1 + k_1 z) = z^{-1} A(1/z)$$

Second Order Lattice:



$$f_2[n] = f_1[n] + k_2\, g_1[n-1]$$
$$g_2[n] = g_1[n-1] + k_2\, f_1[n]$$
Time domain

Substituting $f_1[n]$ & $g_1[n]$ into equations above

$$f_2[n] = f_0[n] + k_1\, g_0[n-1] + k_2\left[g_0[n-2] + k_1 f_0[n-1]\right]$$

$$f_2[n] = x[n] + \underbrace{k_1(1+k_2)}_{a_2[1]}\, x[n-1] + \underbrace{k_2}_{a_2[2]}\, x[n-2]$$

$$a_2[1] = k_1(1+k_2)$$
$$a_2[2] = k_2$$
(2 equations in 2 unknowns)

Solving the above yields

$$k_2 = a_2[2]$$
$$k_1 = \frac{a_2[1]}{1 + a_2[1]}$$

$$A_2(z) = 1 + k_1(1+k_2)z^{-1} + k_2 z^{-2} = \frac{F_2(z)}{X(z)}$$

$$g_2[n] = g_0[n-2] + k_1 f_0[n-1]$$
$$+ k_2 \left\{ f_0[n] + k_1 g_0[n-1] \right\}$$

$$g_2[n] = g_0[n-2] + k_1(1+k_2)\, g_0[n-1]$$
$$+ k_2\, g_0[n]$$

$$\frac{G_2(z)}{X(z)} = k_2 + k_1(1+k_2) z^{-1} + z^{-2}$$
$$= z^{-2}\left(1 + k_1(1+k_2) z + k_2 z^2\right)$$

$$\frac{G_2(z)}{X(z)} = z^{-2} A_2(1/z)$$

General Recursion:

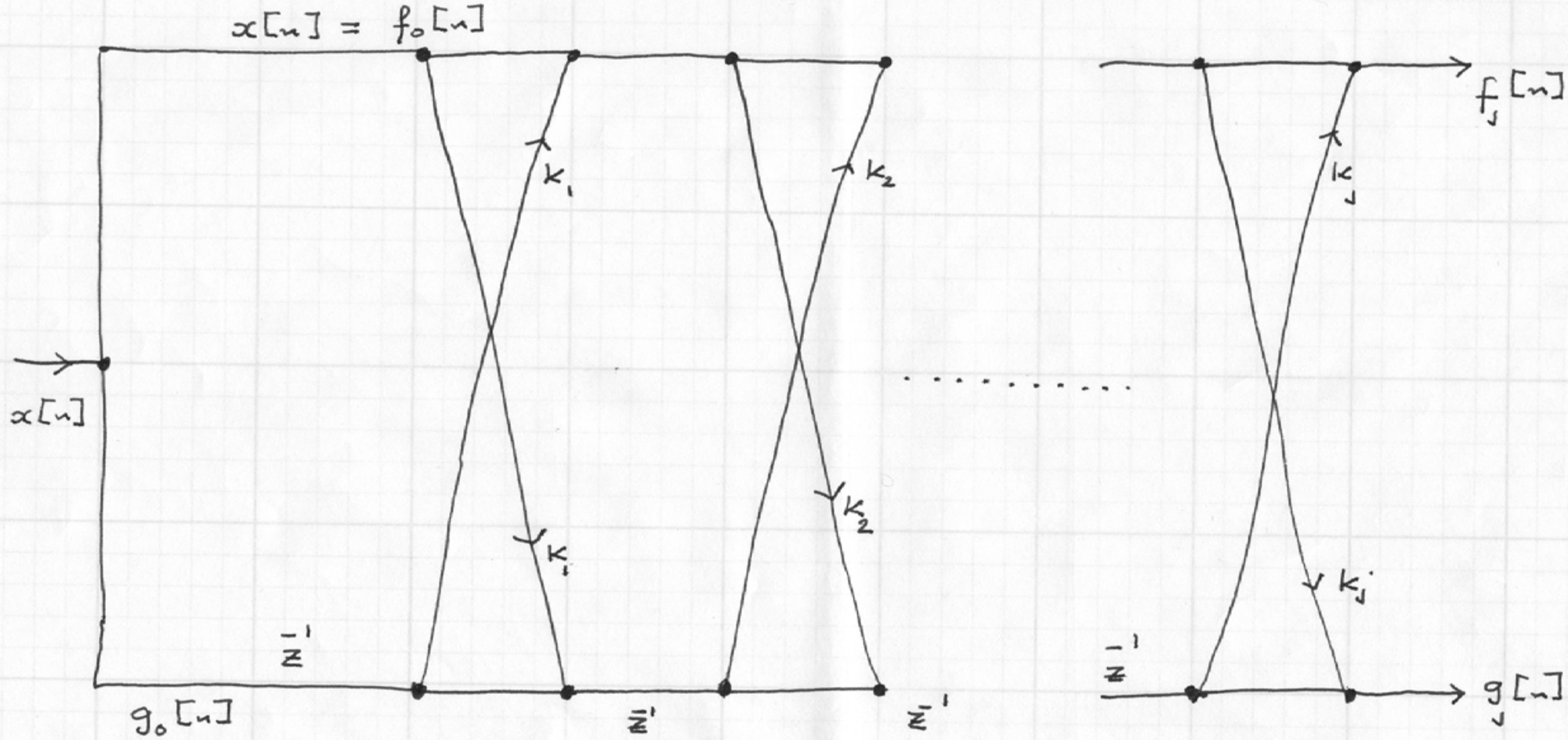
$$f_j[n] = f_{j-1}[n] + k_j\, g_{j-1}[n-1]$$
$$g_j[n] = k_j\, f_{j-1}[n] + g_{j-1}[n-1]$$
$$f_0[n] = g_0[n] = x[n]$$

Z-domain:

$$A_0(z) = \frac{F_0(z)}{X(z)} = 1$$

$$B_0(z) = \frac{G_0(z)}{X(z)} = 1$$

$$\begin{pmatrix} F_j(z) \\ G_j(z) \end{pmatrix} = \begin{pmatrix} 1 & k_j \\ k_j & 1 \end{pmatrix} \begin{pmatrix} F_{j-1}(z) \\ z^{-1} G_{j-1}(z) \end{pmatrix}$$



$$\frac{G_J(z)}{X(z)} = B_J(z) = z^{-J} A_J(z^{-1}) \quad \text{(Reverse prediction polynomial)}$$

Recursion:

$$A_j(z) = A_{j-1}(z) + k_j z^{-1} B_{j-1}(z)$$
$$B_j(z) = k_j A_{j-1}(z) + z^{-1} B_{j-1}(z)$$

$$\begin{pmatrix} A_j(z) \\ B_j(z) \end{pmatrix} = \begin{pmatrix} 1 & k_j \\ k_j & 1 \end{pmatrix} \begin{pmatrix} A_{j-1}(z) \\ z^{-1} B_{j-1}(z) \end{pmatrix}$$

Step down recursion:

$$A_j(z) = A_{j-1}(z) + k_j \left(B_j(z) - k_j A_{j-1}(z)\right)$$

$$A_{j-1}(z) = \frac{1}{1-k_j^2} A_j(z) - \frac{k_j}{1-k_j^2} B_j(z)$$

$$K_m = a_m[m]$$

$$a_{m-1}[0] = 1$$

$$a_{m-1}[k] = \frac{1}{1-k_m^2} a_m[k] - \frac{k_m}{1-k_m^2} b_m[k]$$

$$1 \le k \le m-1$$

- Equations are singular if $|k_m| = 1$
- Root on unit circle
- Factor out root and apply recursion to remainder

IIR Lattice Structures

The system with system function $A_m(z)$ has the following difference equation

$$A_m(z) = \frac{F_m(z)}{X(z)}$$

$$\Rightarrow \quad y[n] = f_m[n] = \sum_{k=0}^{m} a_m[k]\, x[n-k]$$

or

$$y[n] = x[n] + \sum_{k=1}^{m} a_m[k]\, x[n-k]$$

The system with system function $\frac{1}{A_m(z)}$ has the DE representation

$$y[n] = -\sum_{k=1}^{m} a_m[k]\, y[n-k] + x[n]$$

$\Rightarrow$ Roles of $x[n]$ & $y[n]$ are reversed

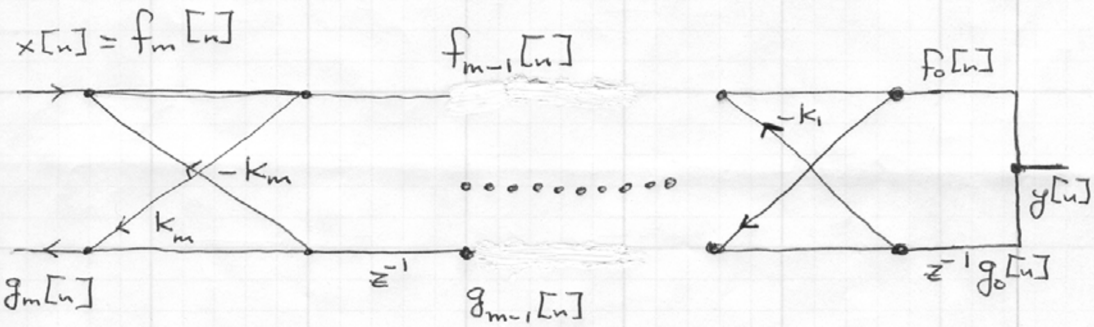
$\Rightarrow$ FIR lattice with input replaced by output will yield an IIR lattice

Lattice Equations

$$f_m[n] = f_{m-1}[n] + k_m g_{m-1}[n-1]$$
$$g_m[n] = g_{m-1}[n-1] + k_m f_{m-1}[n]$$

Reshaped lattice equations

$$f_{m-1}[n] = f_m[n] - k_m g_{m-1}[n-1]$$
$$g_m[n] = g_{m-1}[n-1] + k_m f_{m-1}[n]$$



- Corresponds to a all-pole lattice
- characterised by same set of reflection coefficients
- $$B_m(z) = \frac{G_m(z)}{Y(z)} = z^{-m} A_m(z)$$

has the same all-zero form as in FIR lattice

Modularity:

$$\begin{pmatrix} A_j(z) \\ B_j(z) \end{pmatrix} = \begin{pmatrix} 1 & k_j z^{-1} \\ k_j & z^{-1} \end{pmatrix} \begin{pmatrix} A_{j-1}(z) \\ B_{j-1}(z) \end{pmatrix}$$

$$\begin{pmatrix} A_1(z) \\ B_1(z) \end{pmatrix} = \begin{pmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + k_1 z^{-1} \\ k_1 + z^{-1} \end{pmatrix}$$

$$\begin{pmatrix} A_2(z) \\ A_1(z) \end{pmatrix} = \begin{pmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 + k_1 z^{-1} \\ k_1 + z^{-1} \end{pmatrix}$$

$$A_2(z) = 1 + k_1(1+k_2) z^{-1} + k_2 z^{-2}$$

$$\begin{pmatrix} A_3(z) \\ B_3(z) \end{pmatrix} = \begin{pmatrix} 1 & k_3 z^{-1} \\ k_3 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 + k_1(1+k_2) z^{-1} + k_2 z^{-2} \\ k_2 + k_1(1+k_2) z^{-1} + z^{-2} \end{pmatrix}$$

and this recursion can be continued

- To get to $A_m(z)$ from $A_{m-1}(z)$ you only need to determine k_m

Stability:

$$\prod_{i=1}^{m} r_i = k_m \quad \text{(product of roots)}$$

If system is stable $\Rightarrow |r_i| < 1, \ \forall i$

$$\Rightarrow |k_m| < 1 \quad \text{for } m = 0, \ldots M-1$$

Characterestics:

(a) Lattice structure modular:

$$f_m[n] = f_{m-1}[n] + k_m g_{m-1}[n-1]$$
$$g_m[n] = k_m f_{m-1}[n] + g_{m-1}[n]$$

Addition of an extra stage does not affect earlier stages

(b) $$f_m[n] = f_{m-1}[n] + k_m g_{m-1}[n-1]$$

(Prediction error signal)

Minimizing

$E\{|f_m[n]|^2\}$ w.r.s.t k_m yields.

$$k_m = - \frac{E\left\{ f_{m-1}[n]\, g^*_{m-1}[n-1] \right\}}{E\{|g_{m-1}[n-1]|^2\}}$$

$\Rightarrow$ k_m is a normalized correlation (PARCOR)

$\Rightarrow$ $|k_m| < 1$ if variances are finite

$\Rightarrow$ Monitoring stability convenient

$\Rightarrow$ $E\{|f_m[n]|^2\} < E\{|f_{m-1}[n]|^2\}$ ....

(c) Robust to finite-wordlength effects.

(d) Requires only m reflection coefficients $\{k_i\}_{i=1}^{m}$ to implement system functions $\{A_i\}_{i=1}^{m}$

(e) Convenient for slow fading communications applications (monitoring of stability easy)

and guaranteed stability for slow non stationary environments

Example:

$$K_1 = \frac{1}{4}, \quad K_2 = \frac{1}{2}, \quad k_3 = \frac{1}{3}$$

$$A_0(z) = B_0(z) = 1$$

$$A_1(z) = 1 + k_1 z^{-1} = 1 + \tfrac{1}{4} z^{-1}$$

$$B_1(z) = z^{-1}(1 + \tfrac{1}{4} z) = \tfrac{1}{4} + z^{-1}$$

$$\begin{pmatrix} A_2(z) \\ B_2(z) \end{pmatrix} = \begin{pmatrix} 1 & \tfrac{1}{2} \\ \tfrac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} A_1(z) \\ z^{-1} B_1(z) \end{pmatrix}$$

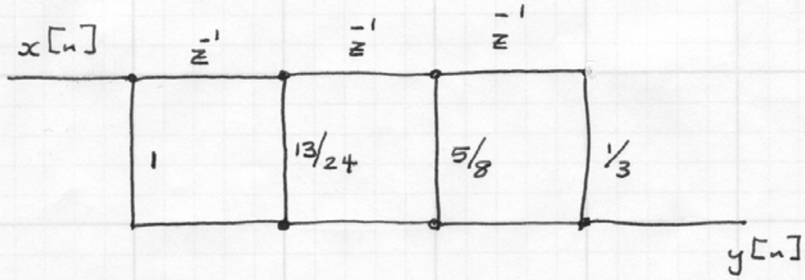
$$\begin{aligned} A_2(z) &= A_1(z) + \frac{1}{2} z^{-1} B_1(z) \\ &= 1 + \frac{1}{4} z^{-1} + \frac{1}{2} z^{-1} (\tfrac{1}{4} + z^{-1}) \\ &= 1 + \frac{3}{8} z^{-1} + \tfrac{1}{2} z^{-2} \end{aligned}$$

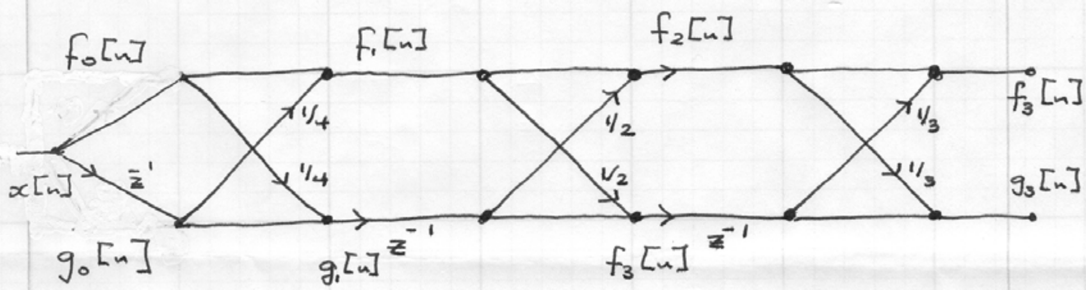
$$\begin{aligned} B_2(z) &= \tfrac{1}{2} (1 + \tfrac{1}{4} z^{-1}) + z^{-1} (\tfrac{1}{4} + z^{-1}) \\ &= \tfrac{1}{2} + \tfrac{3}{8} z^{-1} + z^{-2} \end{aligned}$$

$$A_3(z) = A_2(z) + \tfrac{1}{3} z^{-1} B_2(z)$$

$$A_3(z) = 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-1} \left(\frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2} \right)$$

$$A_3(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \tfrac{1}{3} z^{-3}$$





Step Down Recursion!

$$A_3(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$$

$a_3(0) = 1$
$a_3(1) = 13/24$
$a_3(2) = 5/8$
$a_3(3) = k_3 = 1/3$

$j = 2$:
$a_2(0) = 1$
$B_3(z) = \frac{1}{3} + 5/8\, z^{-1} + 13/24\, z^{-2} + z^{-3}$

$$A_2(z) = \frac{1}{8/9} A_3(z) - \frac{1/3}{8/9} B_3(z)$$
$$= 1 + 3/8\, z^{-1} + 1/2\, z^{-2} \Rightarrow k_2 = 1/2$$

Order $J = 1$:

$$A_1(z) = \frac{1}{1 - 1/4} A_2(z) - \frac{1/2}{1 - 1/4} B_2(z)$$

$$= \frac{4}{3}\left(1 + 3/8\, z^{-1} + \frac{1}{2} z^{-2}\right) - \frac{2}{3}\left(\frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}\right)$$

$$= 1 + \frac{6}{24} z^{-1}$$

$$\Longrightarrow \quad K_1 = \frac{6}{24} = \frac{1}{4}$$