On Least Squares Inversion

A problem of importance that we will see appear often in optimal estimation is the concept of least-squares inversion. Here we will look at the basic idea behind the least-squares estimation hoopla. Consider a system of m linear equations in n unknowns:

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Our goal is to look at two specific situations of interest:

1. $\text{rank}(\mathbf{A}) = m$, i.e., $\mathbf{A}$ is of full row-rank.
2. $\text{rank}(\mathbf{A}) = n$, i.e., $\mathbf{A}$ is of full column-rank.

Let us first look at the case where the system has full row-rank. The rectangular system described above does not have a unique solution because $\mathbf{b} \notin \text{range}(\mathbf{A})$. If the matrix $\mathbf{A}$ is of full row-rank then the matrix $\mathbf{A}\mathbf{A}^H$ is self-adjoint and furthermore:

$$\text{rank}(\mathbf{A}\mathbf{A}^H) = \text{rank}(\mathbf{A}) = m.$$

Upon making the substitution $\mathbf{x} = \mathbf{A}^H\mathbf{y}$ we obtain a modified system of linear equations:

$$\mathbf{A}\mathbf{A}^H\mathbf{y} = \mathbf{b}.$$

Since the matrix $\mathbf{A}\mathbf{A}^H$ is invertible we can obtain the solution to the modified system as:

$$\mathbf{y} = (\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{b}.$$

Substituting this solution back into the original system we obtain:

$$\mathbf{x}_{\text{ls}} = \mathbf{A}^H\mathbf{y} = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{b} = \mathbf{A}_r^\dagger\mathbf{b}.$$

The matrix $\mathbf{A}_r^\dagger$ is refered to as the *least squares* generalized right inverse. Note that this is a true right inverse because:

$$\mathbf{A}\mathbf{A}_r^\dagger = \mathbf{A}\mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1} = \mathbf{I}.$$

Note further that the following relation holds:

$$\mathbf{A}\mathbf{A}_r^\dagger\mathbf{A} = \mathbf{A}.$$

On the other hand the matrix $\mathbf{P} = \mathbf{A}_r^\dagger\mathbf{A}$ is idempotent since:

$$\mathbf{P}^2 = \left[\mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{A}\right]\left[\mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{A}\right] = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{A} = \mathbf{P}.$$

This relation implies that the eigenvalues of $\mathbf{P}$ are either zero or one since:

$$\mathbf{P}^2\mathbf{v} = \mathbf{P}\mathbf{v} = \lambda\mathbf{v} = \lambda^2\mathbf{v}.$$

The matrix $\mathbf{P}$ is in particular the projection matrix that projects vectors on to the row space of the matrix $\mathbf{A}$. Further note that this projection matrix is a self-adjoint projection matrix, i.e.,

$$\mathbf{P}^H = \left(\mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{A}\right)^H = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-H}\mathbf{A} = \mathbf{A}^H((\mathbf{A}\mathbf{A}^H)^H)^{-1}\mathbf{A} = \mathbf{P}.$$

The projection matrix onto the orthogonal complement is given by:

$$\tilde{\mathbf{P}} = \mathbf{I} - \mathbf{P} = \mathbf{I} - \mathbf{A}_r^{\dagger}\mathbf{A} = \mathbf{I} - \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{A}.$$

This constitutes the projection onto the null space of the matrix $\mathbf{A}$ because for any $\mathbf{b} \in \mathbf{R}^m$:

$$\mathbf{A}\tilde{\mathbf{P}}\mathbf{b} = \mathbf{A}(\mathbf{I} - \mathbf{A}_r^{\dagger}\mathbf{A})\mathbf{b} = (\mathbf{A} - \mathbf{A}\mathbf{A}_r^{\dagger}\mathbf{A})\mathbf{b} = \mathbf{0}\mathbf{b}.$$

The error in the least-squares solution is evaluated as:

$$\mathbf{e}_1 = \tilde{\mathbf{P}}\mathbf{b} = \mathbf{b} - \mathbf{A}^{\dagger}\mathbf{A}\mathbf{b}.$$

The energy in the least-squares error is therefore given by:

$$\epsilon_1^2 = ||e_1||_2^2 = \mathbf{b}^H\tilde{\mathbf{P}}^H\tilde{\mathbf{P}}\mathbf{b}.$$

When the matrix $\mathbf{A}$ has full column-rank then the following observations hold:

- The matrix $\mathbf{A}^H\mathbf{A}$ is self-adjoint.
- The matrix $\mathbf{A}^H\mathbf{A}$ is of full-rank, i.e., $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^H\mathbf{A}) = n$.

Multiplying both sides of the original equation system by $\mathbf{A}^H$ yields:

$$\mathbf{A}^H\mathbf{A}\mathbf{x}_{\text{ls}} = \mathbf{A}^H\mathbf{b}.$$

The solution to this modified system is given by:

$$\mathbf{x}_{\text{ls}} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{b} = \mathbf{A}_l^{\dagger}\mathbf{b}.$$

The matrix $\mathbf{A}_l^{\dagger}$ is refered to as the least-squares generalized left-inverse because:

$$\mathbf{A}_l^{\dagger}\mathbf{A} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{A} = \mathbf{I}, \quad \mathbf{A}\mathbf{A}_l^{\dagger}\mathbf{A} = \mathbf{A}.$$

The projection matrix $\mathbf{P}_r$ onto the range space of the matrix $\mathbf{A}$ is given by:

$$\mathbf{P}_r = \mathbf{A}\mathbf{A}_l^{\dagger} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H.$$

The corresponding projection operator onto the left-null space of the matrix $\mathbf{A}$ is given by:

$$\tilde{\mathbf{P}}_{\text{ln}} = \mathbf{I} - \mathbf{A}\mathbf{A}_l^{\dagger} = \mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

Furthermore, this is indeed a projection onto the left-null space because for $\mathbf{b} \in \mathbf{R}^n$:

$$\mathbf{b}^H\mathbf{P}_{\text{ln}}\mathbf{A} = \mathbf{b}^H(\mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H)\mathbf{A} = \mathbf{b}^H(\mathbf{A} - \mathbf{A}) = \mathbf{0}.$$

The error in the corresponding least-squares estimate is given via:

$$\mathbf{e}_2 = \tilde{\mathbf{P}}_{\text{ln}}\mathbf{b}, \quad \epsilon_2^2 = \mathbf{b}^H\tilde{\mathbf{P}}_{\text{ln}}^H\tilde{\mathbf{P}}_{\text{ln}}\mathbf{b}.$$