Notes on Upsampling and Downsampling

In the section on the Nyquist sampling theorem we saw that the operation of sampling a finite energy, bandlimited continuous–time signal at points $t = nT_s$ aliases the spectrum of the signal according to

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T_s}\right). \tag{1}$$

In this section we will study the effects of sampling the signal $x_c(t)$ above the Nyquist rate, i.e, *oversampling* or below the Nyquist rate, i.e, *undersampling*.

Upsampling or Zero Insertion

Let $x[n]$ be the sequence obtained by sampling the continuous–time, finite energy, bandlimited signal $x_c(t)$ at the Nyquist rate, i.e, $x[n] = x_c(nT_s)$. Let $y[n]$ be the sequence obtained by sampling the signal $x_c(t)$ at L times the Nyquist rate, i.e., $\tilde{T}_s = \frac{\pi}{L\Omega_m}$ and

$$y[n] = x_c(n\tilde{T}_s) = x_c\left(\frac{nT_s}{L}\right).$$

The first observation is that sampling the signal at L times the Nyquist rate introduces spectral redundancy into the so obtained sequence because we can see that copies of spectrum $X_c(\Omega)$ are now shifted by $L\frac{2\pi}{T_s}$ as described by the alias sum formula:

$$X_s(\Omega) = \frac{1}{\tilde{T}_s} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{\tilde{T}_s}\right) = \frac{L}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - kL\frac{2\pi}{T_s}\right).$$

The discrete–time equivalent of oversampling is the upsampling system. Consider the discrete–time upsampling system with input $x[n]$ and output $y[n]$ related through

$$\boxed{y[n] = x\left[\frac{n}{L}\right] = \sum_{p=-\infty}^{\infty} x[p]\delta[n - pL].} \tag{2}$$

The output sequence $y[n]$ is essentially the samples $x[n]$ with $(L-1)$ zeros inserted inbetween them. The spectral redundancy described in the frequency-domain translates into zero samples in the time-domain. Let us now look at the upsampling system in the Z-domain. Using the definition of the Z-transform we have

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} x[p]\delta[n-pL]z^{-n}.$$

After interchanging the order of the summations and using the delay property of the Z-transform we can see that

$$\boxed{Y(z) = \sum_{p=-\infty}^{\infty} x[p]z^{-pL} = X(z^L).} \tag{3}$$

Evaluated on the unit circle, i.e, $z = e^{j\omega}$ this translates to a compression of the frequency axis, i.e,

$$Y(e^{j\omega}) = X(e^{j\omega L}). \tag{4}$$

Note that now $Y(e^{j\omega})$, the DTFT of the sequence $y[n]$ is now periodic with a period of $\frac{2\pi}{L}$ as seen from:

$$Y\left(e^{j\left(\omega + r\frac{2\pi}{L}\right)}\right) = X\left(e^{j\left(\omega + r\frac{2\pi}{L}\right)L}\right) = X(e^{j\omega L}) = Y(e^{j\omega}).$$

The DTFT of the sequence $y[n]$, $Y(e^{j\omega})$, therefore has L compressed copies of the DTFT of $x[n]$, $X(e^{j\omega})$, in the range $\omega \in [-\pi, \pi]$ as opposed to just one. This phenomenon is refered to as *spectral imaging* or compression. This is a direct consequence of the spectral redundancy introduced by oversampling the continuous–time signal $x_c(t)$.

Subsampling or Downsampling

Let us now look at the operation of undersampling the continuous time signal $x_c(t)$ by sampling the signal using $\tilde{T}_s = MT_s$. The resultant sequence is denoted $y_d[n]$. Obviously by sampling at a rate less that the Nyquist rate we are loosing information due to spectral aliasing described by

$$X_s(\Omega) = \frac{1}{\tilde{T}_s}\sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{\tilde{T}_s}\right) = \frac{1}{MT_s}\sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{MT_s}\right).$$

The discrete–time representation of undersampling is given by the downsampling system with input $x[n]$ and output $y_d[n]$ related through:

$$y_d[n] = x[Mn] = \sum_{p=-\infty}^{\infty} x[Mp]\delta[n-p].$$

This operation essentially throws out $(M-1)$ samples of the input and retains only those samples with indices that are integer multiples of the downsampling factor M. The loss of information is rather obvious in the time–domain. Let us now redirect our attention to the Z-domain. Let us first begin by defining a intermediate sequence $y[n]$ via

$$y[n] = x[n], \quad n = 0, \pm M, \pm 2M, \dots, \pm rM.$$

The output sequence $y_d[n]$ is therefore a sampled version of this intermediate sequence, i.e,

$$y_d[n] = y[Mn] = x[Mn].$$

This operation can be modeled using the sampling operation and Fourier series representation as :

$$y[n] = x[n]p_M[n] = x[n] \sum_{i=-\infty}^{\infty} \delta[n - iM] = x[n] \frac{1}{M} \sum_{i=0}^{M-1} W_M^{-in},$$

where the notation $W_M = e^{-j\frac{2\pi}{M}}$ has been used. The Z-transform of the downsampled sequence $y_d[n]$ in terms of the Z-transform of this intermediate sequence $y[n]$ is :

$$Y_d(z) = \sum_{n=-\infty}^{\infty} y[Mn]z^{-n} = \sum_{k=-\infty}^{\infty} y[k]z^{-\frac{k}{M}} = Y(z^{\frac{1}{M}}).$$

Relating $Y(z)$ to $X(z)$ we have :

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \frac{1}{M} \sum_{i=0}^{M-1} W_M^{-in} z^{-n}.$$

After rearranging the sums we have

$$Y(z) = \frac{1}{M} \sum_{i=0}^{M-1} \sum_{n=-\infty}^{\infty} x[n](zW_M^i)^{-n} = \frac{1}{M} \sum_{i=0}^{M-1} X(zW_M^i).$$

Relating $Y_d(z)$ to $Y(z)$ and finally to $X(z)$ we have:

$$\boxed{Y_d(z) = Y(z^{\frac{1}{M}}) = \frac{1}{M} \sum_{i=0}^{M-1} X(z^{\frac{1}{M}} W_M^i).}$$

If we further evaluate the Z-transform on the unit circle, i.e, $z = e^{j\omega}$ we have the DTFT relation:

$$\boxed{Y_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right).} \tag{5}$$

Undersampling of the signal or $x_c(t)$ or downsampling of the sequence $x[n]$ therefore, results in spectral aliasing of $X(e^{j\omega})$. In other words the spectral equivalent to throwing away $(M-1)$ samples of $x[n]$ is spectral aliasing.

Eq. (5) is instructive because it is similar in form to the alias sum equation in Eq. (1) that results when we sample the continuous–time signal $x_c(t)$. The subsampling process can therefore be thought of as a two-step process. In essence in the subsampling process we are going from a continuous grid in the variable t to discrete grid of samples at $t = nT_s$ and then to a more sparse discrete grid at $t = nMT_s$.