Notes on Decimation and Interpolation

In the section on upsampling and downsampling were the discrete–time equivalent of oversampling and undersampling. In this section we will look at the operations of decimation and interpolation that combine discrete–time filtering with upsampling or downsampling operations.

Decimation or Band Isolation

We saw in the previous section that just the operation of downsampling by a factor of M corresponds to throwing out $(M-1)$ samples in the time–domain or spectral aliasing in the frequency domain. If instead we filtered the Nyquist–rate sampled signal $x[n]$ first with an ideal lowpass filter defined via:

$$H_d(e^{j\omega}) = \begin{cases} 1 & \omega \in [-\frac{\pi}{M}, \frac{\pi}{M}] \\ 0 & \text{otherwise} \end{cases}$$

to produce the signal $y[n]$ before downsampling the signal then we have:

$$y_d[n] = y[Mn] = \sum_{k=-\infty}^{\infty} x[k]h_d[Mn-k].$$

The lowpass filtering effectively bandlimits the signal $x[n]$ to the frequency band $\omega \in [-\frac{\pi}{M}, \frac{\pi}{M}]$. This filtering operation will now permit us to perform a M fold downsampling of $y[n]$ without loosing the information in $y[n]$. In the frequency domain the operation of decimation can be written as:

$$Y_d(z) = \frac{1}{M} \sum_{p=0}^{M-1} H_d(z^{\frac{1}{M}} W_M^i) X(z^{\frac{1}{M}} W_M^i).$$

The output signal $y_d[n]$ contains essentially a band of the signal with frequencies in the range $\omega \in [-\frac{\pi}{M}, \frac{\pi}{M}]$. This operation will be useful in the analysis section of filter banks and design of multirate filter banks systems where we need to analyze specific spectral characterestics of the signal. This is done by separating the frequency content of the signal into different bands.

In the preceeding analysis we have assumed that the filter used to spectrally bandlimit the signal is an ideal filter. In practice this will not be the case and the filter will be designed to have a cut-off frequency of $\omega_o = \frac{\pi}{M}$.

Interpolation

We also saw that the operation of oversampling a continuous–time signal $x_c(t)$ by a factor of L introduces spectral redundancy and corresponds to the discrete–time system of upsampling where $(L-1)$ zeroes were inserted between samples. These zero samples do not carry extra or meaningful information.

To obtain additional samples that are meaningful we need to fill in the zero samples created in the upsampling operation by interpolating between the non-zero samples. This can be accomplished by filtering the upsampled signal $y[n]$ with an ideal lowpass filter given by:

$$H_i^{(1)}(e^{j\omega}) = \begin{cases} L & \omega \in [-\frac{\pi}{L}, \frac{\pi}{L}] \\ 0 & \text{otherwise.} \end{cases}$$

The pertinent results in the time-domain are:

$$y_i[n] = y[n] * h_i^{(1)}[n] = \sum_{p=-\infty}^{\infty} x[p] \text{Sinc}\left(\frac{n}{L} - p\right).$$

This operation of upsampling followed by filtering, therefore, corresponds to sinc interpolation between the Nyquist samples $x[n]$. Of course other alternatives for the interpolation filter can also be used. Linear interpolation specifically corresponds to the use of a Bartlett window of the form

$$h_i^{(2)}[n] = \begin{cases} 1 - \frac{|n|}{L} & , \ n \leq L \\ 0 & , \ n > L. \end{cases}$$

In the frequency domain this corresponds to the operation of filtering the upsampled signal $y[n]$ using $H_i(e^{j\omega})$ of the form:

$$H_i^{(2)}(e^{j\omega}) = \left(\frac{1}{L}\right)\left(\frac{\sin^2\left(\frac{L\omega}{2}\right)}{\sin^2\left(\frac{\omega}{2}\right)}\right).$$

Comparing this to the ideal lowpass sinc–interpolation filter we see that both of them have a DC gains of L, i.e.,

$$H_i^{(1)}(1) = H_i^{(2)}(1) = L.$$

The zeroes of the linear interpolation filter response are at the frequencies ω_k given by:

$$\omega_k = \left(\frac{2\pi}{L}\right)k, \quad k \in \mathbf{I}.$$

In either case the overall frequency domain relation between the Nyquist samples $x[n]$ and the interpolated version $y_i[n]$ is:

$$Y_i(e^{j\omega}) = H_i(e^{j\omega})X(e^{j\omega L}).$$