Polyphase Decomposition

The multirate operations of decimation and interpolation that were introduced in the previous section will now be used to decompose any system function $H(z)$ into its *polyphase representation*. Consider a discrete–time LTI system with impulse response $h[n]$ and system function $H(z)$. Let us begin by first defining the sequences $h_k[n], 0 \leq k \leq M-1$ via:

$$h_k[n] = \begin{cases} h[n+k] & \text{if } n = rM, r \in I \\ 0 & \text{otherwise.} \end{cases}$$

The impulse response $h[n]$ can be decomposed in terms of these sequences $h_k[n]$ via:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k].$$

The k^{th} *polyphase component* of the impulse response $h[n]$ with respect to the integer M denoted $e_k[n]$ is defined via:

$$e_k[n] = h[Mn+k] = h_k[Mn]. \tag{1}$$

We shall see that system function $H(z)$ can also be expanded in terms of these polyphase components. But first let us define the k^{th} polyphase filter via its system function:

$$E_k(z) = \sum_{n=-\infty}^{\infty} e_k[n] z^{-n} = \sum_{n=-\infty}^{\infty} h[Mn+k] z^{-n}. \tag{2}$$

If we express the integer n in terms of its modulo M representation (remainder theorem) as $n = Mr + p, -\infty \leq r \leq \infty, 0 \leq p \leq M-1$ then the system function $H(z)$ can then be written in the form:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{p=0}^{M-1} \sum_{r=-\infty}^{\infty} h[Mr+p] z^{-Mr} z^{-p}.$$

Relating the inner sum to the polyphase filter we have:

$$\boxed{H(z) = \sum_{p=0}^{M-1} E_p(z^M) z^{-p}.} \tag{3}$$

This above decomposition of the system function $H(z)$ in terms of the polyphase filters $E_k(z)$ in Eq. (3) is referred to as *type I polyphase* decomposition. It also allows us to define a filter bank structure with M branches where each branch contains the p^{th} polyphase filter $E_p(z)$, a factor M upsampler and the appropriate branch delay z^{-p}.

Another alternative representation can be obtained by permuting the components around via:

$$\boxed{H(z) = \sum_{p=0}^{M-1} R_p(z^M) z^{-(M-1-l)}}.$$

This representation is called the *Type II polyphase* representation. Note that the type II components are just permutations of the type I components, i.e., $R_l(z) = E_{M-1-l}(z)$. Our motivation for studying these polyphase structures is the efficiency and computational savings that these structures can provide. Specifically:

1. The polyphase filters $E_i(z)$ have a fewer number of filter coefficients in comparison to the system function $H(z)$.

2. They are more efficient from a resource utilization standpoint because the filter structure is not inactive at any time instant.

3. Lower data rates as we see in branches of the filterbank in turn amount to faster computation and reduced complexity for data transfer.

Note further that this polyphase decomposition is not unique in that it depends on the integer M.